

A GEOMETRIC THEORY FOR HYPERGRAPH MATCHING

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ABSTRACT. We develop a theory for the existence of perfect matchings in hypergraphs under quite general conditions. Informally speaking, the obstructions to perfect matchings are geometric, and are of two distinct types: ‘space barriers’ from convex geometry, and ‘divisibility barriers’ from arithmetic lattice-based constructions. To formulate precise results, we introduce the setting of simplicial complexes with minimum degree sequences, which is a generalisation of the usual minimum degree condition. We determine the essentially best possible minimum degree sequence for finding an almost perfect matching. Furthermore, our main result establishes the stability property: under the same degree assumption, if there is no perfect matching then there must be a space or divisibility barrier. This allows the use of the stability method in proving exact results. Besides recovering previous results, we apply our theory to the solution of two open problems on hypergraph packings: the minimum degree threshold for packing tetrahedra in 3-graphs, and Fischer’s conjecture on a multipartite form of the Hajnal-Szemerédi Theorem. Here we prove the exact result for tetrahedra and the asymptotic result for Fischer’s conjecture; since the exact result for the latter is technical we defer it to a subsequent paper.

1. INTRODUCTION

Hypergraph matchings¹ provide a general framework for many important Combinatorial problems. Two classical open problems of this nature are the question of whether there exist designs of arbitrary strength, and Ryser’s conjecture that every Latin square of odd order has a transversal; these are both equivalent to showing that some particular hypergraphs have perfect matchings. Furthermore, matchings are also an important tool for many practical questions, such as the ‘Santa Claus’ allocation problem (see [1]). However, while Edmonds’ algorithm [10] provides an efficient means to determine whether a graph has a perfect matching, the decision problem is NP-complete in k -graphs for $k \geq 3$ (it is one of Karp’s original 21 NP-complete problems [18]). Thus we do not expect a nice characterisation, so we concentrate on natural sufficient conditions for finding a perfect matching. One natural hypergraph parameter that is widely considered in the literature is the *minimum degree* $\delta(G)$, which is the largest number m such that every set of $k - 1$ vertices is contained in at least m edges of G . What is the minimum degree threshold for finding a perfect matching? (We assume that $k|n$, where $n = |V|$.) For graphs ($k = 2$) a straightforward greedy argument shows that the threshold is $n/2$ (or one can deduce it from Dirac’s theorem [9], which states that the same threshold even gives a Hamilton cycle). The general case was a long-standing open problem, finally resolved by Rödl, Ruciński and Szemerédi [37], who determined the threshold precisely for large n : it is $n/2 - k + C$, where $C \in \{1.5, 2, 2.5, 3\}$ depends on arithmetic properties of k and n . We refer the reader to the survey by Rödl and Ruciński [36] for many further results on degree conditions for matchings and Hamilton cycles in hypergraphs.

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¹A *hypergraph* G consists of a vertex set V and an edge set E , where each edge $e \in E$ is a subset of V . We say G is a k -*graph* if every edge has size k . A *matching* M is a set of vertex disjoint edges in G . We call M *perfect* if it covers all of V .

To motivate the results of this paper it is instructive to consider the extremal examples for the minimum degree problems. Consider a graph G_1 on n vertices whose edges are all pairs incident to some set S of size $n/2 - 1$. Then $\delta(G_1) = n/2 - 1$, and G_1 has no perfect matching, as each edge of a matching M uses a vertex in S , so $|M| \leq |S|$; we say that G_1 has a *space barrier*. Now suppose $n/2$ is odd and consider a graph G_2 on n vertices consisting of two disjoint complete graphs of size $n/2$. Then $\delta(G_2) = n/2 - 1$, and G_2 has no perfect matching, but for the different reason that edges have even size; we say that G_2 has a *divisibility barrier*. While these two examples are equally good for graphs, for general k -graphs a separation occurs. For G_1 we take a k -graph whose edges are all k -tuples incident to some set S of size $n/k - 1$; this satisfies $\delta(G_1) = n/k - 1$. For G_2 we take a k -graph whose edges are all k -tuples that have an even size intersection with some set S such that $n/2 - 1 \leq |S| \leq (n + 1)/2$ and $|S|$ is odd; this satisfies $\delta(G_1) = n/2 - k + C$, being one of the extremal constructions in the result of [37] mentioned above. One should also note that space is a robust obstruction to matchings, in that the size of a maximum matching in G_1 may be decreased by decreasing $|S|$, whereas divisibility is not robust, in the sense that any construction similar to G_2 has an almost perfect matching. In fact, Rödl, Ruciński and Szemerédi [38] showed that the minimum degree threshold for a matching of size $n/k - t$ with $t \geq k - 2$ is $n/k - t$. Thus we see a sharp contrast between the thresholds for perfect matching and almost perfect matching.

The main message of this paper is that space and divisibility are the determining factors for perfect matchings in hypergraphs under quite general conditions, and that these factors are inherently geometric. The first part of the geometric theory was anticipated by a result on fractional perfect matchings in [38] that we generalise here. The key point is that fractional perfect matchings correspond to representing a constant vector as a convex combination of edge vectors, so non-existence of fractional perfect matchings can be exploited in geometric form via separating hyperplanes. Furthermore, the fractional problem has bearing on the original problem through standard ‘regularity machinery’, which converts a fractional solution into a matching covering all but $o(n)$ vertices for large n . The second part of the theory is to understand when the number of uncovered vertices can be reduced to a constant independent of n . The idea here can only be properly explained once we have described the regularity embedding strategy, but a brief summary is as follows. Firstly, the $o(n)$ uncovered vertices arise from imbalances created when converting from the fractional to the exact solution. Secondly, the possible ‘transferrals’ of imbalances can be expressed geometrically by defining an appropriate polyhedron and testing for a ball around the origin of some small constant radius; this can also be understood in terms of separating hyperplanes. Thus the first two parts of the theory are problems of convex geometry, which correspond to space barriers. The third part of the theory concerns divisibility barriers, which determine when the number of uncovered vertices can be reduced from a constant to zero. We will see that in the absence of space barriers, perfect matchings exist except in hypergraphs that are structurally close to one of a certain class of arithmetic constructions defined in terms of lattices in \mathbb{Z}^d for some $d \leq k$. Furthermore, since the constructions with space or divisibility barriers do not have perfect matchings, in a vague sense we have ‘the correct theory’, although this is not a precise statement because of additional assumptions. Our theory is underpinned by the ‘strong’ hypergraph regularity theory independently developed by Gowers [15] and Rödl et al. [13, 33, 39, 40], and the recent hypergraph blowup lemma of Keevash [19]. Fortunately,

this part of the argument is mostly covered by existing machinery, so the majority of this paper is devoted to the geometric theory outlined above.

To formulate precise results, we introduce the setting of simplicial complexes with minimum degree sequences, which is a generalisation of the minimum degree condition previously considered. In this setting our main theorems (stated in Section 2.4) give minimum degree sequences that guarantee a perfect matching for hypergraphs that are not close to a lattice construction. These minimum degree sequences are best possible, and furthermore have the ‘stability’ property that, unless the hypergraph is structurally close to a space barrier construction, one can find a perfect matching even with a slightly smaller degree sequence. We defer the statements until we have given the necessary definitions in the next section. For now we want to emphasise the power of this framework by describing its application to the solutions of two open problems on packings. Suppose G is a k -graph on n vertices and H is a k -graph on h vertices, where $h|n$ (we think of h as fixed and n as large). An H -packing in G is a set of vertex-disjoint copies of H inside G ; it is *perfect* if there are n/h such copies, so that every vertex of G is covered. In the case when H is a single edge we recover the notion of (perfect) matchings. As for matchings we have the natural question: what is the minimum degree threshold for finding a perfect H -packing? Even for graphs, this is a difficult question with a long history. One famous result is the Hajnal-Szemerédi theorem [16], which determines the threshold for the complete graph K_r on r vertices: if $r|n$ and $\delta(G) \geq (r-1)n/r$ then G has a K_r -packing, and this bound is best possible. It is interesting to note that this is essentially the same threshold as for the Turán problem [43] for K_{r+1} , i.e. finding a single copy of the complete graph with one more vertex. The perfect packing problem for general graphs H was essentially solved by Kühn and Osthus [25], who determined the threshold for large n up to an additive constant $C(H)$. The precise statement would take us too far afield here, but we remark that the threshold is determined by either space or divisibility barriers, and that the dependence on the chromatic number of H continues a partial analogy with Turán problems. We refer the reader to their survey [26] for further results of this type.

For hypergraph packing problems, the natural starting point is the tetrahedron K_4^3 , i.e. the complete 3-graph on 4 vertices. Here even the asymptotic existence threshold is a long-standing open problem; this is an important test case for general hypergraph Turán problems, which is a difficult area with very few known results (see the survey by Keevash [19]). In light of this, it is perhaps surprising that we are able here to determine the tetrahedron packing threshold for large n , not only asymptotically but precisely. One should note that the two problems are not unrelated; indeed Turán-type problems for the tetrahedron are required when showing that there are no divisibility barriers (but fortunately they are more tractable than the original problem!) The extremal example for the perfect tetrahedron packing problem is by no means obvious, and it was several years after the problem was posed by Abbasi (reported by Czygrinow and Nagle [6]) that Pikhurko [34] provided the optimal construction (we describe it in Section 8). Until recently, the best published upper bounds, also due to Pikhurko [34], were $0.8603\dots n$ for the perfect packing threshold and $3n/4$ for the almost perfect packing threshold. More recent upper bounds for the perfect packing threshold are $4n/5$ by Keevash and Zhao (unpublished) and $(3/4 + o(1))n$ independently by Keevash and Mycroft (earlier manuscripts of this paper) and by Lo and Markström [27] (posted online very recently). It is instructive to contrast the ‘absorbing technique’ used in

the proof of [27] with the approach here; we will make some remarks later to indicate why our methods seem more general and are able to give the exact result, which is as follows.

Theorem 1.1. *There exists n_0 such that if G is a 3-graph on $n \geq n_0$ vertices such that $4 \mid n$ and*

$$\delta(G) \geq \begin{cases} 3n/4 - 2 & \text{if } 8 \mid n \\ 3n/4 - 1 & \text{otherwise,} \end{cases}$$

then G contains a perfect K_4^3 -packing. This minimum degree bound is best possible.

Our second application is to a conjecture of Fischer [12] on a multipartite form of the Hajnal-Szemerédi Theorem. Suppose V_1, \dots, V_k are disjoint sets of n vertices each, and G is a k -partite graph on vertex classes V_1, \dots, V_k (that is, G is a graph on $V_1 \cup \dots \cup V_k$ such that no edge of G has both vertices in the same V_j). Then we define the *partite minimum degree* of G , denoted $\delta^*(G)$, to be the largest m such that every vertex has at least m neighbours in each part other than its own, i.e.

$$\delta^*(G) = \min_{i \in [k]} \min_{v \in V_i} \min_{j \in [k] \setminus \{i\}} |N(v) \cap V_j|,$$

where $N(v)$ denotes the neighbourhood of v . Fischer conjectured that if $\delta^*(G) \geq (k-1)n/k + 1$ then G has a perfect K_k -packing (actually his original conjecture did not include the $+1$, but this stronger conjecture is known to be false). The case $k=2$ of the conjecture is an immediate corollary of Hall's Theorem, whilst the cases $k=3$ and $k=4$ were proved by Magyar and Martin [29] and Martin and Szemerédi [30] respectively. Also, Csaba and Mydlarz [5] proved a weaker version of the conjecture in which the minimum degree condition has an error term depending on k . The following theorem, an almost immediate corollary of our results on hypergraph matchings, gives an asymptotic version of this conjecture for any k .

Theorem 1.2. *For any k and $c > 0$ there exists n such that if G is a k -partite graph with parts V_1, \dots, V_k of size n and $\delta^*(G) \geq (k-1)n/k + cn$, then G contains a perfect K_k -packing.*

In fact, by considering the near-extremal cases of the conjecture using the ‘stability property’ of our main theorem, we are able to prove an exact result in [22], namely that Fischer’s conjecture holds for any sufficiently large n . However, this stability analysis is lengthy and technical, so we prefer to divest this application from the theory developed in this paper.

As mentioned above, we will deduce both Theorems 1.1 and 1.2 from a general framework of matchings in simplicial complexes. These will be formally defined in the next section, but we briefly indicate the connection here. For Theorem 1.1 we consider the ‘clique 4-complex’, with the tetrahedra in G as 4-sets, G as 3-sets, and all smaller sets; for Theorem 1.2 we consider the ‘clique k -complex’, with the j -cliques of G as j -sets for $j \leq k$. In both cases, the required perfect packing is equivalent to a perfect matching using the highest level sets of the clique complex.

Notation. The following notation is used throughout the paper: $[k] = \{1, \dots, k\}$; if X is a set then $\binom{X}{k}$ is the set of subsets of X of size k , and $\binom{X}{\leq k} = \bigcup_{i \leq k} \binom{X}{i}$ is the set of subsets of X of size at most k ; $o(1)$ denotes a term which tends to zero as n tends to infinity; $x \ll y$ means that for every $y > 0$ there exists some $x_0 > 0$ such that the subsequent statement holds for any $x < x_0$ (such statements with more variables are defined similarly). We write

$x = y \pm z$ to mean $y - z \leq x \leq y + z$. Also we denote all vectors in bold font, and their co-ordinates by subscripts of the same character in standard font, e.g. $\mathbf{a} = (a_1, \dots, a_n)$.

2. RESULTS AND EXAMPLES

In this section we state our main theorems on perfect matchings in simplicial complexes. For almost perfect matchings it requires no additional work to obtain more general results that dispense with the ‘downward closure’ assumption. However, for perfect matchings it is more convenient to assume downward closure, which seems to hold in any natural application of our results, so we will stick to simplicial complexes, and make some remarks later on how the approach may be generalised. We also discuss several examples that illustrate various aspects of the theory: space barriers and tightness of the degree condition, lattice-based constructions, and generalisations of previous results.

2.1. Almost perfect matchings. We start with some definitions. We identify a hypergraph H with its edge set, writing $e \in H$ for $e \in E(H)$, and $|H|$ for $|E(H)|$. A k -system is a hypergraph J in which every edge of J has size at most k and $\emptyset \in J$. We refer to the edges of size r in J as r -edges of J , and write J_r to denote the r -graph on $V(J)$ formed by these edges. It may be helpful to think of the r -graphs J_r as different ‘levels’ of the k -system J . A k -complex J is a k -system whose edge set is closed under inclusion, i.e. if $e \in H$ and $e' \subseteq e$ then $e' \in H$. That is, each level of J is supported by the levels below it. For any non-empty k -graph G , we may generate a k -complex G^{\leq} whose edges are any $e \subseteq V(G)$ such that $e \subseteq e'$ for some edge $e' \in G$.

We introduce the following notion of degree in a k -system J . For any edge e of J , the *degree* $d(e)$ of e is the number of $(|e| + 1)$ -edges e' of J which contain e as a subset. (Note that this is *not* the standard notion of degree used in k -graphs, in which the degree of a set is the number of edges containing it.) The *minimum r -degree* of J , denoted $\delta_r(J)$, is the minimum of $d(e)$ taken over all r -edges $e \in J$. So every r -edge of J is contained in at least $\delta_r(J)$ of the $(r + 1)$ -edges of J . Note that if J is a k -complex then $\delta_r(J) \leq \delta_{r-1}(J)$ for each $r \in [k - 1]$. The *degree sequence* of J is $\delta(J) = (\delta_0(J), \delta_1(J), \dots, \delta_{k-1}(J))$.

Our minimum degree assumptions will always be of the form $\delta(J) \geq \mathbf{a}$ pointwise for some vector $\mathbf{a} = (a_0, \dots, a_{k-1})$, i.e. $\delta_i(J) \geq a_i$ for $0 \leq i \leq k - 1$. It is helpful to interpret this ‘dynamically’ as follows: when constructing an edge of J_k by greedily choosing one vertex at a time, there are at least a_i choices for the $(i + 1)$ st vertex (this is the reason for the requirement that $\emptyset \in J$, which we need for the first choice in the process). To see that this generalises the setting of minimum degree in hypergraphs, consider any k -graph G and let J be the k -system on $V(G)$ with $J_k = G$ and complete lower levels $J_i = \binom{V(G)}{i}$, $0 \leq i \leq k - 1$. Then $\delta(J) = (n, n - 1, \dots, n - k + 2, \delta(G))$ is a degree sequence in which the first $k - 1$ co-ordinates are as large as possible, so any minimum degree condition for this k -system J reduces to a minimum degree condition for G . More generally, for any $i \geq 0$, define an i -clique of G to be a set $I \subseteq V(G)$ of size i such that every $K \subseteq I$ of size k is an edge of G (this last condition is vacuous if $i < k$). The *clique r -complex* $J_r(G)$ of G is defined by taking $J_r(G)_i$ to consist of the i -cliques of G for $0 \leq i \leq r$. To see the power in the extra generality of the degree sequence setting, consider the 4-graph of tetrahedra in a 3-graph G . This does not satisfy any minimum degree condition as a 4-graph, as non-edges of G are not contained in any 4-edges; on the other hand, we will see later that a minimum degree condition on G implies a useful minimum degree condition for the clique 4-complex.

Our first theorem on hypergraph matchings gives a sufficient minimum degree sequence for a k -system to contain a matching covering almost all of its vertices.

Theorem 2.1. *Suppose that $1/n \ll \alpha \ll 1/k$, and that J is a k -system on n vertices with*

$$\delta(J) \geq \left(n, \left(\frac{k-1}{k} - \alpha \right) n, \left(\frac{k-2}{k} - \alpha \right) n, \dots, \left(\frac{1}{k} - \alpha \right) n \right).$$

Then J_k contains a matching which covers all but at most $9k^2\alpha n$ vertices of J .

Next we describe a family of examples showing that Theorem 2.1 is best possible up to the constant $9k^2$ (which we do not attempt to optimise), in the sense that there may not be any almost perfect matching if *any* co-ordinate of the degree sequence is substantially reduced.

Construction 2.2. (*Space barriers*) Suppose V is a set of n vertices, $j \in [k-1]$ and $S \subseteq V$. Let $J = J(S, j)$ be the k -complex in which J_i (for $0 \leq i \leq k$) consists of all i -sets in V that contain at most j vertices of S . Note that $J_i = \binom{V}{i}$ is complete for $i \leq j$. We have $\delta_i(J) = n - i$ for $0 \leq i \leq j-1$ and $\delta_i(J) = n - |S| - (i - j)$. Choosing $|S| = \left\lfloor \left(\frac{j}{k} + \alpha \right) n \right\rfloor - k$, we see that the minimum degree condition of Theorem 2.1 is satisfied. However, every edge of J_k has at least $k - j$ vertices in $V \setminus S$, so the maximum matching in J_k has size $\left\lfloor \frac{|V \setminus S|}{k-j} \right\rfloor$, which leaves at least $\alpha n - k$ uncovered vertices.

In these examples we have a space barrier for matchings, which is robust in the sense that the number of uncovered vertices varies smoothly with the minimum degree sequence. Our preliminary stability result is that space barriers are the only obstructions to finding a matching that covers all but a constant number of vertices. Note that we cannot avoid leaving a constant number of uncovered vertices because of divisibility barriers (discussed below). Our structures permit some small imperfections, defined as follows. Suppose G and H are k -graphs on the same set V of n vertices and $0 < \beta < 1$. We say that G is β -contained in H if all but at most βn^k edges of G are edges of H .

Theorem 2.3. *Suppose that $1/n \ll 1/\ell \ll \alpha \ll \beta \ll 1/k$. Let J be a k -complex on n vertices such that*

$$\delta(J) \geq \left(n, \left(\frac{k-1}{k} - \alpha \right) n, \left(\frac{k-2}{k} - \alpha \right) n, \dots, \left(\frac{1}{k} - \alpha \right) n \right).$$

Then J has at least one of the following properties:

- 1 (Matching):** J_k contains a matching that covers all but at most ℓ vertices.
- 2 (Space barrier):** J_k is β -contained in $J(S, j)_k$ for some $j \in [k-1]$ and $S \subseteq V(J)$ with $|S| = \lfloor jn/k \rfloor$.

2.2. Partite systems. We will also require ‘partite’ analogues of our hypergraph matching theorems. A *partition* \mathcal{P} of a set V is a sequence of disjoint sets V_1, \dots, V_k whose union is V ; we refer to the sets V_i as the *parts* of \mathcal{P} , and write $U \in \mathcal{P}$ to mean that U is a part of \mathcal{P} . Note that we consider the partition \mathcal{P} to describe not just the contents of each part but also the order of the parts. We say that the partition \mathcal{P} is *balanced* if each part has the same size.

Let H be a hypergraph, and let \mathcal{P} be a partition of $V(H)$. Then we say a set $S \subseteq V(H)$ is \mathcal{P} -*partite* if it has at most one vertex in any part of \mathcal{P} , and that H is \mathcal{P} -*partite* if every

edge of H is \mathcal{P} -partite. We say that H is r -partite if there exists some partition \mathcal{P} of $V(H)$ into r parts such that H is \mathcal{P} -partite. For r -partite k -systems we introduce the following alternative notion of degree. Let V be a set of vertices, let \mathcal{P} be a partition of V into r parts V_1, \dots, V_r , and let J be a \mathcal{P} -partite k -system on V . For each $0 \leq j \leq k-1$ we define the *partite minimum j -degree* $\delta_j^*(J)$ as the largest m such that any j -edge e has at least m extensions to a $(j+1)$ -edge in any part not used by e , i.e.

$$\delta_j^*(J) := \min_{e \in J_j} \min_{i: e \cap V_i = \emptyset} |\{v \in V_i : e \cup \{v\} \in J\}|.$$

The *partite degree sequence* is $\delta^*(J) = (\delta_0^*(J), \dots, \delta_{k-1}^*(J))$. Note that we suppress the dependence on \mathcal{P} in our notation: this will be clear from the context.

Our next theorem is an analogue of Theorem 2.1 for r -partite k -systems.

Theorem 2.4. *Suppose that $1/n \ll \alpha \ll 1/r \leq 1/k$, and that J is a r -partite k -system on vertex classes each of n vertices with*

$$\delta^*(J) \geq \left(n, \left(\frac{k-1}{k} - \alpha \right) n, \left(\frac{k-2}{k} - \alpha \right) n, \dots, \left(\frac{1}{k} - \alpha \right) n \right).$$

Then J_k contains a matching which covers all but at most $9k^2 r \alpha n$ vertices of J .

To see that this is best possible in the same sense as for Theorem 2.1 we use the natural partite version of Construction 2.2, which shows that there may not be any almost perfect matching if any co-ordinate of the partite degree sequence is substantially reduced.

Construction 2.5. (*Partite space barriers*) Suppose \mathcal{P} partitions a set V into r parts V_1, \dots, V_r of size n . Let $j \in [k-1]$, $S \subseteq V$ and $J = J_r(S, j)$ be the k -complex in which J_i (for $0 \leq i \leq k$) consists of all \mathcal{P} -partite i -sets in V that contain at most j vertices of S . We choose S to have $s = \left\lfloor \left(\frac{j}{k} + \alpha \right) n \right\rfloor$ vertices in each part. Then $\delta_i^*(J) = n$ for $0 \leq i \leq j-1$ and $\delta_i^*(J) = n-s$ for $j \leq i \leq k-1$, so the minimum partite degree condition of Theorem 2.4 is satisfied. However, the maximum matching in J_k has size $\left\lfloor \frac{|V \setminus S|}{k-j} \right\rfloor$, which leaves at least $r(\alpha n - k)$ uncovered vertices.

We also have the following stability result analogous to Theorem 2.3.

Theorem 2.6. *Suppose that $1/n \ll 1/\ell \ll \alpha \ll \beta \ll 1/r \leq 1/k$. Let \mathcal{P} partition a set V into parts V_1, \dots, V_r each of size n , and let J be a \mathcal{P} -partite k -complex on V with*

$$\delta^*(J) \geq \left(n, \left(\frac{k-1}{k} - \alpha \right) n, \left(\frac{k-2}{k} - \alpha \right) n, \dots, \left(\frac{1}{k} - \alpha \right) n \right).$$

Then J has at least one of the following properties:

- 1 (Matching):** J_k contains a matching that covers all but at most ℓ vertices.
- 2 (Space barrier):** J_k is β -contained in $J_r(S, j)_k$ for some $j \in [k-1]$ and $S \subseteq V$ with $\lfloor jn/k \rfloor$ vertices in each V_i , $i \in [r]$.

2.3. Lattice-based constructions. Having described the general form of space barriers, we now turn our attention to divisibility barriers. For this we need the notion of *index vectors*, which will play a substantial role in this paper. Let V be a set of vertices, and let \mathcal{P} be a partition of V into d parts V_1, \dots, V_d . Then for any $S \subseteq V$, the *index vector of S with respect to \mathcal{P}* is the vector $\mathbf{i}_{\mathcal{P}}(S) := (|S \cap V_1|, \dots, |S \cap V_d|)$ in \mathbb{Z}^d . When \mathcal{P} is clear from the context, we write simply $\mathbf{i}(S)$ for $\mathbf{i}_{\mathcal{P}}(S)$. So $\mathbf{i}(S)$ records how many vertices of S are in

each part of \mathcal{P} . Note that $\mathbf{i}(S)$ is well-defined as we consider the partition \mathcal{P} to define the order of its parts.

Construction 2.7. (*Divisibility barriers*) Suppose L is a lattice in \mathbb{Z}^d (i.e. an additive subgroup) with $\mathbf{i}(V) \notin L$, fix any $k \geq 2$, and let G be the k -graph on V whose edges are all k -tuples e with $\mathbf{i}(e) \in L$. For any matching M in G with vertex set $S = \bigcup_{e \in M} e$ we have $\mathbf{i}(S) = \sum_{e \in M} \mathbf{i}(e) \in L$. Since we assumed that $\mathbf{i}(V) \notin L$ it follows that G does not have a perfect matching.

We will now give some concrete illustrations of this construction.

- (1) Suppose $d = 2$ and $L = \langle (-2, 2), (0, 1) \rangle$. Note that $(x, y) \in L$ precisely when x is even. Then by definition, $|V_1|$ is odd, and the edges of G are all k -tuples $e \subseteq V$ such that $|e \cap V_1|$ is even. If $|V| = n$ and $|V_1| \sim n/2$, then we recover the extremal example mentioned earlier for the minimum degree perfect matching problem. The generated k -complex G^{\leq} has $\delta_i(G^{\leq}) \sim n$ for $0 \leq i \leq k-2$ and $\delta_{k-1}(G^{\leq}) \sim n/2$.
- (2) Suppose $d = 3$ and $L = \langle (-2, 1, 1), (1, -2, 1), (1, 0, 0) \rangle$. Note that $(x, y, z) \in L$ precisely when $y = z \pmod 3$. Thus $|V_2| \not\equiv |V_3| \pmod 3$ and the edges of G are all k -tuples $e \subseteq V$ such that $|e \cap V_2| = |e \cap V_3| \pmod 3$. Note that for any $(k-1)$ -tuple $e' \subseteq V$ there is a unique $j \in [3]$ such that $\mathbf{i}(e') + \mathbf{u}_j \in L$, where \mathbf{u}_j is the j th standard basis vector. (We have $j = 1$ if $\mathbf{i}(e')_2 = \mathbf{i}(e')_3 \pmod 3$, $j = 2$ if $\mathbf{i}(e')_2 = \mathbf{i}(e')_3 - 1 \pmod 3$, or $j = 3$ if $\mathbf{i}(e')_2 = \mathbf{i}(e')_3 + 1 \pmod 3$.) If $|V| = n$ and $|V_1|, |V_2|, |V_3| \sim n/3$ then $\delta_i(G^{\leq}) \sim n$ for $0 \leq i \leq k-2$ and $\delta_{k-1}(G^{\leq}) \sim n/3$. Mycroft [32] showed that this construction is asymptotically extremal for a range of hypergraph packing problems. For example, he showed that any 4-graph G on n vertices with $7 \mid n$ and $\delta(G) \geq n/3 + o(n)$ contains a perfect $K_{4,1,1,1}^4$ -packing, where $K_{4,1,1,1}^4$ denotes the complete 4-partite 4-graph with vertex classes of size 4, 1, 1 and 1 respectively; this construction demonstrates that this bound is best possible up to the $o(n)$ error term.

For simplicity we only gave approximate formulae for the degree sequences in these examples, but it is not hard to calculate them exactly. Note that the index vectors of edges in a k -graph belong to the hyperplane $\{\mathbf{x} \in \mathbb{R}^d : \sum_{i \in [d]} x_i = k\}$. The intersection of this hyperplane with the lattice L is a coset of the intersection of L with the hyperplane $\Pi^d = \{\mathbf{x} \in \mathbb{R}^d : \sum_{i \in [d]} x_i = 0\}$. Thus it is most convenient to choose a basis of L that includes a basis of $L \cap \Pi^d$, as in the above two examples. In order for the construction to exist it must be possible to satisfy $\mathbf{i}(V) \notin L$, so L should not contain the lattice $M_k^d = \{\mathbf{x} \in \mathbb{Z}^d : k \mid \sum_{i \in [d]} x_i\}$. We say that L is *complete* if $M_k^d \subseteq L$, otherwise L is *incomplete*. Assuming that L contains some \mathbf{v} with $\sum v_i = k$, an equivalent formulation of completeness is that $L \cap \Pi^d = \mathbb{Z}^d \cap \Pi^d$, since this is equivalent to $L \cap (\Pi^d + \mathbf{v}) = \mathbb{Z}^d \cap (\Pi^d + \mathbf{v}) = \{\mathbf{x} \in \mathbb{Z}^d : \sum_{i \in [d]} x_i = k\}$, and so to $M_k^d \subseteq L$.

There is a natural notion of minimality for lattice-based constructions. We say that L is *minimal* if it does not contain any difference of standard basis vectors, i.e. a vector of the form $\mathbf{u}_i - \mathbf{u}_j$ with $i \neq j$. If L is not minimal, then the construction for L can be reformulated using a lattice of smaller dimension. Without loss of generality we may consider the case that $\mathbf{u}_{d-1} - \mathbf{u}_d \in L$. Then we replace \mathcal{P} by the partition of V into $d-1$ parts obtained by combining V_{d-1} and V_d , and replace L by the lattice L' consisting of all $\mathbf{x} \in \mathbb{Z}^{d-1}$ such that $(x_1, \dots, x_{d-1}, 0) \in L$. We remark that a minimal lattice in \mathbb{Z}^d has index at least d as a subgroup of \mathbb{Z}^d . To see this, note that the cosets $L + (\mathbf{u}_1 - \mathbf{u}_i)$, $1 \leq i \leq d$ must be distinct,

otherwise there is some $\mathbf{x} \in \mathbb{Z}^d$ and $i, j \in [d]$ for which $\mathbf{x} = \mathbf{v} + \mathbf{u}_1 - \mathbf{u}_i = \mathbf{v}' + \mathbf{u}_1 - \mathbf{u}_j$ with $\mathbf{v}, \mathbf{v}' \in L$, which gives $\mathbf{u}_i - \mathbf{u}_j = \mathbf{v} - \mathbf{v}' \in L$, contradicting minimality.

2.4. Perfect matchings. For our main theorems we specialise to the setting of simplicial complexes, which will simplify some arguments and is sufficiently general for our applications. Under the same minimum degree assumption that is tight for almost perfect matchings, we refine the results above, by showing that either we have a perfect matching, or we have a structural description for the complex: it either has a space barrier or a divisibility barrier. It is remarkable that this rigidity emerges from the purely combinatorial degree sequence assumption, and that it exhibits these two very different phenomena, one tied to convex geometry and the other to integer lattices.

Our structures permit some small imperfections, defined as follows. Recall that we say that a k -graph G is β -contained in a k -graph H if all but at most βn^k edges of G are edges of H . Also, given a partition \mathcal{P} of V into d parts, we define the β -robust edge lattice $L_{\mathcal{P}}^{\beta}(G) \subseteq \mathbb{Z}^d$ to be the lattice generated by all vectors $\mathbf{v} \in \mathbb{Z}^d$ such that there are at least βn^k edges $e \in G$ with $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{v}$. Recall that we call a lattice $L \subseteq \mathbb{Z}^d$ complete if $L \cap \Pi^d = \mathbb{Z}^d \cap \Pi^d$, where $\Pi^d = \{\mathbf{x} \in \mathbb{R}^d : \sum_{i \in [d]} x_i = 0\}$. Recall also that the space barrier constructions $J(S, j)$ were defined in Section 2.1. Now we can state our first main theorem.

Theorem 2.8. *Suppose that $1/n \ll \alpha \ll \mu, \beta \ll 1/k$ and that $k \mid n$. Let J be a k -complex on n vertices such that*

$$\delta(J) \geq \left(n, \left(\frac{k-1}{k} - \alpha \right) n, \left(\frac{k-2}{k} - \alpha \right) n, \dots, \left(\frac{1}{k} - \alpha \right) n \right).$$

Then J has at least one of the following properties:

- 1 (Matching):** J_k contains a perfect matching.
- 2 (Space barrier):** J_k is β -contained in $J(S, j)_k$ for some $j \in [k-1]$ and $S \subseteq V(J)$ with $|S| = jn/k$.
- 3 (Divisibility barrier):** There is some partition \mathcal{P} of $V(J)$ into $d \leq k$ parts of size at least $\delta_{k-1}(J) - \mu n$ such that $L_{\mathcal{P}}^{\mu}(J_k)$ is incomplete.

Our second main theorem is a partite version of the previous result. Recall that the space barrier constructions in this setting are described in Construction 2.5. We also need to account for the original partition when classifying edge lattices as follows. Suppose \mathcal{P} is a partition of V into d parts (V_1, \dots, V_d) that refines a partition \mathcal{P}' of V . Let $L_{\mathcal{P}\mathcal{P}'} \subseteq \mathbb{Z}^d \cap \Pi^d$ be the lattice generated by all differences of basis vectors $\mathbf{u}_i - \mathbf{u}_j$ for which V_i, V_j are contained in the same part of \mathcal{P}' . We say that a lattice $L \subseteq \mathbb{Z}^d$ is *complete with respect to \mathcal{P}'* if $L_{\mathcal{P}\mathcal{P}'} \subseteq L \cap \Pi^d$, otherwise we say that L is *incomplete with respect to \mathcal{P}'* .

Theorem 2.9. *Suppose that $1/n \ll \alpha \ll \mu, \beta \ll 1/r \leq 1/k$. Let \mathcal{P}' partition a set V into parts V_1, \dots, V_r each of size n , where $k \mid rn$. Suppose that J is a \mathcal{P}' -partite k -complex with*

$$\delta^*(J) \geq \left(n, \left(\frac{k-1}{k} - \alpha \right) n, \left(\frac{k-2}{k} - \alpha \right) n, \dots, \left(\frac{1}{k} - \alpha \right) n \right).$$

Then J has at least one of the following properties:

- 1 (Matching):** J_k contains a perfect matching.
- 2 (Space barrier):** J_k is β -contained in $J_r(S, j)_k$ for some $j \in [k-1]$ and $S \subseteq V$ with $\lfloor jn/k \rfloor$ vertices in each V_i , $i \in [r]$.

3 (Divisibility barrier): *There is some partition \mathcal{P} of $V(J)$ into $d \leq kr$ parts of size at least $\delta_{k-1}^*(J) - \mu n$ such that \mathcal{P} refines \mathcal{P}' and $L_{\mathcal{P}}^\mu(J_k)$ is incomplete with respect to \mathcal{P}' .*

2.5. Further results. Theorems 2.8 and 2.9 can be applied to a variety of matching and packing problems in graphs and hypergraphs. Indeed, in this section we give a short deduction of Theorem 1.2 from Theorem 2.9, whilst in Section 8 we use Theorems 2.8 to prove Theorem 1.1. We can also recover and find new variants of existing results. For example, consider the result of Rödl, Ruciński and Szemerédi [37] on the minimum degree threshold for a perfect matching in a k -graph. Their proof proceeds by a stability argument, giving a direct argument when the k -graph is close to an extremal configuration, and otherwise showing that even a slightly lower minimum degree is sufficient for a perfect matching. Our results give a new proof of stability under a much weaker degree assumption. In the following result, we only assume that the minimum degree of G is a bit more than $n/3$, and show that if there is no perfect matching then G is almost contained in an extremal example. In fact, with a bit more work (and a stronger degree assumption) we can show that G is contained in an extremal example (see Theorem 8.21). This requires some technical preliminaries, so we postpone the proof for now.

Theorem 2.10. *Suppose $1/n \ll b, c \ll 1$, $k \geq 3$ and G is a k -graph on n vertices with $\delta(G) \geq (1/3 + c)n$ and no perfect matching. Then there is a partition of $V(G)$ into parts V_1, V_2 of size at least $\delta(G)$ and $a \in \{0, 1\}$ so that all but at most bn^k edges e of G have $|e \cap V_1| = a \pmod 2$.*

Proof. Introduce constants μ, β with $1/n \ll \mu, \beta \ll b, c$. Let J be the (clique) k -complex with $J_k = G$ and J_i complete for $i < k$. Suppose that J_k does not have a perfect matching. Then one of options 2 and 3 must hold in Theorem 2.8. First consider option 2, that J_k is β -contained in $J(S, j)_k$ for some $j \in [k-1]$ and $S \subseteq V(J)$ of size jn/k . So $|S| < (k-1)n/k$, and by averaging there is a $(k-1)$ -tuple $T \subseteq V(G) \setminus S$ that is contained in at most $k^k k! \beta n$ edges of $G \setminus J(S, j)_k$. Any edge e of G containing T has at most one vertex in S , so can only appear in $J(S, j)_k$ in the case $j = 1$. Thus we have $d(T) < n/k + k^k k! \beta n < (1/3 + c)n$, contradicting the minimum degree assumption. Now consider option 3, that there is some partition \mathcal{P} of $V(J)$ into parts of size at least $\delta(G) - \mu n$ such that $L_{\mathcal{P}}^\mu(J_k)$ is incomplete. Since $\delta(G) - \mu n > n/3$ there must be 2 such parts, and $L_{\mathcal{P}}^\mu(J_k) \cap \Pi^2$ is generated by $(-t, t)$ for some $t \geq 2$. We cannot have $t > 2$, as then neither $(k-2, 2)$ nor $(k-1, 1)$ lie in $L_{\mathcal{P}}^\mu(J_k)$, and so the at least $(n/3)^{k-1}/(k-1)!$ $(k-1)$ -tuples of index $(k-2, 1)$ are contained in the at most $2\mu n^k$ edges of these two indices. By averaging some such $(k-1)$ -tuple would then be contained in at most $3^k k! \mu n^k$ edges of G , contradicting the minimum degree assumption. So we must have $t = 2$, and the result follows. \square

We now present the deduction of Theorem 1.2 from Theorem 2.9.

Proof of Theorem 1.2. Introduce constants μ, β with $1/n \ll \mu, \beta \ll c$. Let J be the clique k -complex of G . Then $\delta_i(J) \geq (k-i)n/k + icn$ for $0 \leq i \leq k-1$. Thus we can apply Theorem 2.9; we will show that options 2 and 3 cannot hold. First consider option 2, that J_k is β -contained in $J_k(S, j)_k$ for some $j \in [k-1]$ and $S \subseteq V(J)$ with $|S \cap V_i| = jn/k$ for each $i \in [k]$. The minimum degree sequence provides many ordered $(k-j+1)$ -cliques within $V(J) \setminus S$: when greedily choosing the vertices one at a time there are at least $(k-j-i)n/k + icn$

choices for the $(i+1)$ st vertex, so we obtain at least $c(n/k)^{k-j+1}$ such cliques. Then we can extend each such ordered clique to an ordered k -clique with no restrictions on the choices, so that there are at least $(k-i)n/k + icn$ choices for the $(i+1)$ st vertex. This gives at least $c(n/k)^k$ ordered k -cliques, each of which has at most $j-1$ vertices in S , so is not contained in $J_k(S, j)_k$. This is a contradiction if $k!\beta < c/k^k$.

Now consider option 3, that there is some partition \mathcal{P} of $V(J)$ into parts of size at least $\delta_{k-1}^*(J) - \mu n$ such that \mathcal{P} refines the partition \mathcal{P}' of V into V_1, \dots, V_k and $L_{\mathcal{P}}^\mu(J_k)$ is incomplete with respect to \mathcal{P}' . We may assume that $L_{\mathcal{P}}^\mu(J_k)$ is minimal; recall that this means that it does not contain any difference of standard basis vectors $\mathbf{u}_i - \mathbf{u}_j$ with $i \neq j$. Consider a part of \mathcal{P}' that is refined in \mathcal{P} , without loss of generality it is V_1 , and vertices $x_1, x'_1 \in V_1$ in different parts U_1, U'_1 of \mathcal{P} . We can greedily construct many sequences $x_i \in V_i$, $2 \leq i \leq k$ such that $x_1 x_2 \dots x_k$ and $x'_1 x_2 \dots x_k$ are both k -cliques: since $\delta^*(G) \geq (k-1)n/k + cn$ there are at least $(k-i)n/k + icn$ choices for x_i , so at least $c(n/k)^{k-1}$ such sequences. We can repeat this for all choices of $x_1 \in U_1$ and $x'_1 \in U'_1$; there are at least $\delta_{k-1}^*(J) - \mu n > n/k$ choices for each. Since $c \gg \mu$, there is some choice of parts $U_i \subseteq V_i$, $2 \leq i \leq k$ of \mathcal{P} such that we obtain at least μn^k cliques intersecting U_i , $i \in [k]$, and at least μn^k cliques intersecting U'_1 and U_i , $2 \leq i \leq k$. However, this contradicts minimality of $L_{\mathcal{P}}^\mu(J_k)$. Thus Theorem 2.9 implies that J_k has a perfect matching, as required. \square

2.6. Outline of the proofs. The ideas of our arguments can be roughly organised into the following three groups: regularity, transferrals, applications. Most of the new ideas in this paper pertain to transferrals, but to set the scene we start with regularity. The Szemerédi Regularity Lemma [42] has long been a powerful tool in graph theory. In combination with the blowup lemma of Komlós, Sárközy and Szemerédi [24] it has seen many applications to embeddings of spanning subgraphs (see [26]). Recent developments in hypergraph regularity theory have opened the way towards obtaining analogous results for hypergraphs: the decomposition theory (among other things) was developed independently by Gowers [15] and by Rödl et al. [13, 33, 40, 39], and the blowup lemma by Keevash [19]. Roughly speaking, the decomposition theory allows us to approximate a k -system J on n vertices by a ‘reduced’ k -system R on m vertices, where m depends on the accuracy of the approximation, but is independent of n . The vertices of R correspond to the parts in some partition of $V(J)$ into ‘clusters’ of equal size, after moving a small number of vertices to an exceptional set V_0 . The edges of R correspond to groups of clusters for which the restriction of the appropriate level of J is well-approximated by a ‘dense regular’ hypergraph. Furthermore, R inherits from J approximately the same proportional minimum degree sequence. As mentioned earlier, this part of the machinery allows us to reduce the almost perfect matching problem to finding a fractional solution. If J has an extra $o(n)$ in its minimum degree sequence then it is a relatively simple problem in convex geometry to show that R_k has a fractional perfect matching M ; moreover, in the absence of a space barrier we can find M even without this extra $o(n)$. Then we partition the clusters of J in proportion to the weights of edges in M , so that each non-zero edge weight in M is associated to a dense regular k -partite k -graph with parts of equal size (adding a small number of vertices to V_0). It is then straightforward to find almost perfect matchings in each of these k -partite k -graphs, which together constitute an almost perfect matching in J_k .

To find perfect matchings, we start by taking a regularity decomposition and applying the almost perfect matching result in the reduced system. We remove any uncovered clusters,

adding their vertices to the exceptional set V_0 , so that we have a new reduced system R with a perfect matching M . We also transfer a small number of vertices from each cluster to V_0 so that the edges of M now correspond to dense regular k -partite k -graphs that have perfect matchings (rather than almost perfect matchings). In fact, these k -graphs have the stronger property of ‘robust universality’: even after deleting a small number of additional vertices, we can embed any bounded degree subhypergraph. (Up to some technicalities not mentioned here, this is the form in which we apply the hypergraph blowup lemma.) Next we greedily find a partial matching that covers V_0 , and also removes some vertices from the clusters covered by M , where we take care not to destroy robust universality. Now the edges of M correspond to k -partite k -graphs that are robustly universal, but have slightly differing part sizes. To find a perfect matching, we will remove another partial matching that balances the part sizes and does not destroy robust universality. Then the edges of M will now correspond to k -partite k -graphs that have perfect matchings, and together with the two partial matchings these constitute a perfect matching in J_k .

Transferrals come into play in the step of removing a partial matching to balance the part sizes. Given clusters U and U' and $b \in \mathbb{N}$, a b -fold (U, U') -transferral in (R, M) consists of a pair (T, T') of multisets of edges, with $T \subseteq R$, $T' \subseteq M$ and $|T| = |T'|$, such that every cluster is covered equally often by T and T' , except that U is covered b times more in T than in T' , and U' is covered b times more in T' than in T . Given such a pair (T, T') for which U is too large and U' is too small for our perfect matching strategy, we can reduce the imbalance by b ; to achieve this we choose a partial matching in J with edges corresponding to each edge of T (disjoint from all edges chosen so far), and we note for future reference that the perfect matchings corresponding to edges of M chosen in the final step will use one fewer edge corresponding to each edge of T' . We say that (R, M) is (B, C) -irreducible if for any U, U' there is a b -fold (U, U') -transferral of size c for some $b \leq B$ and $c \leq C$ (here we are sticking to the non-partite case for simplicity). Irreducibility allows us to reduce the cluster imbalances to a constant independent of n , and so find a matching in J_k covering all but a constant number of vertices. Furthermore, it can be expressed in terms of the following geometric condition, alluded to in the introduction. Let $X = X(R, M) = \{\chi(e) - \chi(e') : e \in R, e' \in M\}$, where $\chi(S)$ denotes the characteristic vector of a set $S \subseteq V(R)$. The required condition is that the convex hull of X should contain a ball of some small constant radius centred at the origin. Thus irreducibility becomes a question of convex geometry. It is also the second point at which space barriers come into play, as our minimum degree assumption implies the existence of such a ball about the origin in the absence of a space barrier. This leads to our first stability result: under the same minimum degree sequence assumption on J needed for a matching in J_k that is almost perfect (i.e. covers all but $o(n)$ vertices), we can in fact find a matching in J_k that covers all but a constant number of vertices, unless J is structurally close to a space barrier construction.

For perfect matchings, we need more precise balancing operations, namely simple transferrals, by which we mean 1-fold transferrals. We can still use general transferrals for most of the balancing operations, but we require simple transferrals to make the cluster sizes precisely equal. We introduce transferral digraphs $D_\ell(R, M)$, where there is an edge from U to U' if and only if (R, M) contains a simple (U, U') -transferral of size at most ℓ . If we now assume for simplicity that J is a k -complex (rather than just a k -system), then it is immediate that every vertex in $D_1(R, M)$ has outdegree at least $\delta_{k-1}^*(R)$. We will prove a structure result for digraphs with linear minimum outdegree, which when combined with

irreducibility gives a partition \mathcal{P} of $V(R)$ into a constant number of parts, such that there are simple transferrals between any two vertices in the same part. This is the point at which divisibility barriers come into play. If the robust edge lattice of R_k with respect to \mathcal{P} is incomplete then J_k is structurally close to a divisibility barrier construction. Otherwise, for any pair of parts $P_i, P_j \in \mathcal{P}$ we have some simple (U_i, U_j) -transferrals with $U_i \in P_i$ and $U_j \in P_j$, and the robustness of the edge lattice gives enough of these transferrals to make the cluster sizes precisely equal. Thus we obtain the full stability theorem: under the same minimum degree sequence assumption on J needed for a matching in J_k that is almost perfect, we can find a perfect matching in J_k , unless J is structurally close to a space or divisibility barrier construction.

3. GEOMETRIC MOTIFS

In this section we prove or cite various lemmas that underpin the convex geometry in our theory, and also demonstrate the connection with fractional perfect matchings in hypergraphs. For the classical results cited we refer the reader to the book by Schrijver [41]. We begin with Carathéodory's theorem, in a slightly unusual form. Given points $\mathbf{x}_1, \dots, \mathbf{x}_s \in \mathbb{R}^d$, we define their *convex hull* as

$$CH(\mathbf{x}_1, \dots, \mathbf{x}_s) := \left\{ \sum_{j \in [s]} \lambda_j \mathbf{x}_j : \lambda_j \in [0, 1], \sum_{j \in [s]} \lambda_j = 1 \right\}.$$

Theorem 3.1 (Carathéodory's Theorem). *Suppose $X \subseteq \mathbb{R}^d$ and $\mathbf{x} \in CH(X)$. Then there are $\lambda_1, \dots, \lambda_r \geq 0$ and $\mathbf{x}_1, \dots, \mathbf{x}_r \in X$ such that*

- (a) $\sum_{j \in [r]} \lambda_j = 1$,
- (b) *the vectors $\mathbf{x}_j - \mathbf{x}_r$ are linearly independent for each $j \in [r-1]$, and*
- (c) $\mathbf{x} = \sum_{j \in [r]} \lambda_j \mathbf{x}_j$.

Note that condition (b) implies that $r \leq d+1$. Indeed, a more standard statement of Theorem 3.1 has the condition $r \leq d+1$ in place of (b); this is commonly proved by first proving our formulation of the theorem. This means that we may write any point in $CH(X)$ as a positive linear combination of a small number of members of X . The following proposition gives conditions under which we can arrange that all of the coefficients λ_j are rationals of small denominator. We say that a real number x is q -rational if we can write $x = a/b$ for integers a, b with $1 \leq b \leq q$. We let \mathbb{Q}_q^d denote the set of points $\mathbf{x} \in \mathbb{R}^d$ such that every coordinate of \mathbf{x} is q -rational. Also, given $\mathbf{x} \in \mathbb{R}^d$ and $r \geq 0$ we let $B^d(\mathbf{x}, r) = \{\mathbf{z} : \|\mathbf{z} - \mathbf{x}\| \leq r\}$ denote the ball of radius r centred at \mathbf{x} (we sometimes drop the dimension superscript when this is clear from the context).

Proposition 3.2. *Suppose that $1/q' \ll 1/q, 1/d, 1/k$, and let X be a subset of $\mathbb{Q}_q^d \cap B^d(\mathbf{0}, 2k)$. Then for any $\mathbf{x} \in CH(X) \cap \mathbb{Q}_q^d$ there exist $\lambda_1, \dots, \lambda_r \geq 0$ and $\mathbf{x}_1, \dots, \mathbf{x}_r \in X$ such that $r \leq d+1$, each λ_j is q' -rational, $\sum_{j \in [r]} \lambda_j = 1$, and $\mathbf{x} = \sum_{j \in [r]} \lambda_j \mathbf{x}_j$.*

Proof. By Theorem 3.1 we may choose $z_1, \dots, z_r \geq 0$ and $\mathbf{x}_1, \dots, \mathbf{x}_r \in X$ such that $\sum z_j = 1$, the vectors $\mathbf{x}_j - \mathbf{x}_r$ are linearly independent for each $j \in [r-1]$ (and hence $r \leq d+1$) and $\mathbf{x} = \sum_{j \in [r]} z_j \mathbf{x}_j$. We can write $\mathbf{x} = A\mathbf{z} + \mathbf{x}_r$, where A is the d by $r-1$ matrix whose columns are the vectors $\mathbf{x}_i - \mathbf{x}_r$, $i \in [r-1]$, and $\mathbf{z} = (z_1, \dots, z_{r-1})^\top$. Note that A has rank $r-1$, since its columns are linearly independent. Let $\mathbf{v}_1, \dots, \mathbf{v}_d$ be the row

vectors of A . Then we can choose $S \subseteq [d]$ of size $r - 1$ such that the row vectors \mathbf{v}_j , $j \in S$ are linearly independent. Let B be the $r - 1$ by $r - 1$ square matrix with rows \mathbf{v}_j for $j \in S$. Then $\det(B) \neq 0$. Also, since every entry of A has absolute value at most $4k$, we have $\det(B) \leq (r - 1)!(4k)^{r-1}$. (A better bound is available from Hadamard's inequality, but it suffices to use this crude bound which follows by estimating each term in the expansion of the determinant.) Now we write $\mathbf{x}' - \mathbf{x}'_r = B\mathbf{z}$, where \mathbf{x}' is the restriction of \mathbf{x} to the coordinates $j \in S$, and \mathbf{x}'_r is defined similarly. Then $\mathbf{z} = B^{-1}(\mathbf{x}' - \mathbf{x}'_r)$, and so every coordinate of \mathbf{z} can be expressed as a fraction with denominator at most $q^2 \det(B) \leq q'$, as required. \square

Next we need the classical theorem on the equivalence of vertex and half-plane representations of convex polytopes. This is commonly known as the Weyl-Minkowski theorem; it is also implied by results of Farkas. Given points $\mathbf{v}_1, \dots, \mathbf{v}_r \in \mathbb{R}^d$, we define their *positive cone* as

$$PC(\{\mathbf{v}_1, \dots, \mathbf{v}_r\}) := \left\{ \sum_{j \in [r]} \lambda_j \mathbf{v}_j : \lambda_1, \dots, \lambda_r \geq 0 \right\}.$$

The *Minkowski sum* of two sets $A, B \subseteq \mathbb{R}^d$ is $A + B = \{a + b : a \in A, b \in B\}$.

Theorem 3.3 (Weyl-Minkowski Theorem). *Let $P \subseteq \mathbb{R}^d$. Then the following statements are equivalent.*

- (i) $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_j \cdot \mathbf{x} \geq b_j \text{ for all } j \in [s]\}$ for some $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{R}^d$ and $b_1, \dots, b_s \in \mathbb{R}$.
- (ii) $P = CH(X) + PC(Y)$ for some finite sets $X, Y \subseteq \mathbb{R}^d$.

An important case of Theorem 3.3 is when $P = PC(Y)$ for some finite set $Y \subseteq \mathbb{R}^d$. Then we can write $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_j \cdot \mathbf{x} \geq 0 \text{ for all } j \in S\}$ for some $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{R}^d$, since we have $\mathbf{0} \in P$, and if $\mathbf{x} \in P$ then $2\mathbf{x} \in P$. The following result of Farkas follows.

Lemma 3.4 (Farkas' Lemma). *Suppose $\mathbf{v} \in \mathbb{R}^d \setminus PC(Y)$ for some finite set $Y \subseteq \mathbb{R}^d$. Then there is some $\mathbf{a} \in \mathbb{R}^n$ such that $\mathbf{a} \cdot \mathbf{y} \geq 0$ for every $\mathbf{y} \in Y$ and $\mathbf{a} \cdot \mathbf{v} < 0$.*

Our next result exploits the discrete nature of bounded integer polytopes. First we need a convenient description for the faces of a polytope. A *face* of a polytope P is the intersection of P with a set \mathcal{H} of hyperplanes, such that for each $H \in \mathcal{H}$, P is contained in one of the closed halfspaces defined by H . More concretely, consider a finite set $X \subseteq \mathbb{R}^d$ with $\mathbf{0} \in X$. By Theorem 3.3 we can write $CH(X) = \{\mathbf{x} : \mathbf{a}_j \cdot \mathbf{x} \geq b_j \text{ for all } j \in [s]\}$ for some $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{R}^d$ and $b_1, \dots, b_s \in \mathbb{R}$. Let $S = \{j \in [s] : b_j = 0\}$,

$$\Pi_0^X = \{\mathbf{x} : \mathbf{a}_j \cdot \mathbf{x} = 0 \text{ for all } j \in S\} \text{ and } F_0^X = CH(X) \cap \Pi_0^X.$$

Then F_0^X is the (unique) minimum face of $CH(X)$ containing $\mathbf{0}$. Note that in the extreme cases, F_0^X could be all of $CH(X)$, or just the single point $\mathbf{0}$. The following result gives a lower bound for the distance of $\mathbf{0}$ from the boundary of F_0^X .

Lemma 3.5. *Suppose that $0 < \delta \ll 1/k, 1/d$ and $\mathbf{0} \in X \subseteq \mathbb{Z}^d \cap B^d(\mathbf{0}, 2k)$. Then $\Pi_0^X \cap B^d(\mathbf{0}, \delta) \subseteq CH(X)$.*

Proof. For each X with $\mathbf{0} \in X \subseteq \mathbb{Z}^d \cap B^d(\mathbf{0}, 2k)$ we fix a representation $CH(X) = \{\mathbf{x} : \mathbf{a}_j^X \cdot \mathbf{x} \geq b_j^X \text{ for all } j \in [s^X]\}$ such that $b_j^X \in \mathbb{R}$ and $\mathbf{a}_j^X \in \mathbb{R}^d$ with $\|\mathbf{a}_j^X\| = 1$ for $j \in [s^X]$. Since $\mathbf{0} \in X$ we must have $b_j^X \leq 0$ for every j and X . Note that there are only finitely many possible choices of X , and each s^X is finite and depends only on X . Choosing δ sufficiently small, we may assume that for any j and X either $b_j^X = 0$ or $b_j^X < -\delta$. Now

fix any such set X , and write $F_0^X = CH(X) \cap \Pi_0^X$, where $S^X = \{j \in [s^X] : b_j^X = 0\}$ and $\Pi_0^X = \{\mathbf{x} : \mathbf{a}_j^X \cdot \mathbf{x} = 0 \text{ for all } j \in S^X\}$. Consider any $\mathbf{x} \in \Pi_0^X \setminus CH(X)$. Since $\mathbf{x} \in \Pi_0^X$, \mathbf{x} satisfies all constraints $\mathbf{a}_j^X \cdot \mathbf{x} \geq b_j^X$ for $CH(X)$ with $b_j^X = 0$. Since $\mathbf{x} \notin CH(X)$, \mathbf{x} must fail some constraint $\mathbf{a}_j^X \cdot \mathbf{x} \geq b_j^X$ for $CH(X)$ with $b_j^X \neq 0$. It follows that $\mathbf{a}_j^X \cdot \mathbf{x} < b_j^X < -\delta$. Applying the Cauchy-Schwartz inequality, we have $\delta < |\mathbf{a}_j^X \cdot \mathbf{x}| \leq \|\mathbf{a}_j^X\| \|\mathbf{x}\| = \|\mathbf{x}\|$, as required. \square

Now we will demonstrate the connection between convex geometry and fractional perfect matchings, and how these can be obtained under our minimum degree sequence assumption. A *fractional perfect matching* in a hypergraph G is an assignment of non-negative weights to the edges of G such that for any vertex v , the sum of the weights of all edges incident to v is equal to 1. We also require the following notation, which will be used throughout the paper. For any $S \subseteq [n]$, the *characteristic vector* $\chi(S)$ of S is the vector in \mathbb{R}^n given by

$$\chi(S)_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S. \end{cases}$$

If G has n vertices, then by identifying $V(G)$ with $[n]$ we may refer to the characteristic vector $\chi(S)$ of any $S \subseteq V(G)$. Whilst $\chi(S)$ is then dependent on the chosen identification of $V(G)$ with $[n]$, the effect of changing this identification is simply to permute the vectors $\chi(S)$ for each $S \subseteq V(G)$ by the same permutation, and all properties we shall consider will be invariant under this isomorphism. So we will often speak of $\chi(S)$ for $S \subseteq V(G)$ without specifying the identification of $V(G)$ and $[n]$. Then a fractional perfect matching is an assignment of weights $w_e \geq 0$ to each $e \in G$ such that $\sum_{e \in G} w_e \chi(e) = \mathbf{1}$; throughout this paper $\mathbf{1}$ denotes a vector of the appropriate dimension in which every co-ordinate is 1. Thus G has a fractional perfect matching precisely when $\mathbf{1}$ belongs to the positive cone $PC(\chi(e) : e \in G)$. The next lemma shows that this holds under a similar minimum degree sequence assumption to that considered earlier. We include the proof for the purpose of exposition, as a similar argument later in Lemma 7.2 will simultaneously generalise both this statement and a multipartite version of it.

Lemma 3.6. *Suppose that $k \mid n$, and that J is a k -system on n vertices with*

$$\delta(J) \geq (n, (k-1)n/k, (k-2)n/k, \dots, n/k).$$

Then J_k admits a fractional perfect matching.

Proof. Suppose for a contradiction that J_k does not admit a fractional perfect matching. As noted above, this means that $\mathbf{1} \notin PC(\chi(e) : e \in J_k)$. Then by Farkas' Lemma, there is some $\mathbf{a} \in \mathbb{R}^n$ such that $\mathbf{a} \cdot \mathbf{1} < 0$ and $\mathbf{a} \cdot \chi(e) \geq 0$ for every $e \in J_k$. Let v_1, \dots, v_n be the vertices of J , and let a_1, \dots, a_n be the corresponding coordinates of \mathbf{a} , with the labels chosen so that $a_1 \leq a_2 \leq \dots \leq a_n$. For any sets S and S' of k vertices of J , we say that S *dominates* S' if we may write $S = \{v_{i_1}, \dots, v_{i_k}\}$ and $S' = \{v_{j_1}, \dots, v_{j_k}\}$ so that $j_\ell \leq i_\ell$ for each $\ell \in [k]$. Note that if S dominates S' then $\mathbf{a} \cdot \chi(S') \leq \mathbf{a} \cdot \chi(S)$; this follows from the fact that the coordinates of \mathbf{a} are increasing.

For each $j \in [n/k]$, let S_j be the set $\{v_j, v_{j+n/k}, v_{j+2n/k}, \dots, v_{j+(k-1)n/k}\}$. Then the sets S_j partition $V(J)$, so $\sum_{j \in [n/k]} \mathbf{a} \cdot \chi(S_j) = \mathbf{a} \cdot \mathbf{1}$. We claim that there is some edge $e \in J_k$ which is dominated by every S_j . To see this, we let $d_1 = 1$, then apply the minimum degree sequence condition on J to choose d_2, \dots, d_k greedily so that for each $j \in [k]$ we have $\{v_{d_1}, \dots, v_{d_j}\} \in J$

and $d_j \leq (j-1)n/k + 1$. Then $e := \{v_{d_1}, \dots, v_{d_k}\}$ is dominated by S_j for each $j \in [n/k]$. We therefore have

$$0 \leq \sum_{j \in [n/k]} \mathbf{a} \cdot \chi(e) \leq \sum_{j \in [n/k]} \mathbf{a} \cdot \chi(S_j) = \mathbf{a} \cdot \mathbf{1} < 0,$$

which is a contradiction. \square

4. TRANSFERRALS

To motivate the results of the next two sections, we start by recalling the proof strategy discussed in Section 2.6. Hypergraph regularity theory (presented in Section 6) will enable us to approximate our original k -system by a reduced k -system. To avoid confusion over notation, we should emphasise that all of our transferral results will be applied when J is equal to the reduced system, rather than the original system. We will also have a perfect matching M in J_k , whose edges represent k -tuples of clusters in which we would be able to find a perfect matching, if we were able to make the cluster sizes equal. The role of transferrals is to achieve this by removing a small partial matching. While they are motivated by the proof strategy, the definition and analysis of transferrals does not require any regularity theory (apart from one technical lemma in the next section). Suppose that J is a k -graph on $[n]$ and M is a perfect matching in J ; for convenience we call the pair (J, M) a *matched k -graph*. Recall that $\chi(e) \in \mathbb{R}^n$ denotes the characteristic vector of $e \subseteq [n]$. If T is a multiset of subsets of $[n]$ we write $\chi(T) = \sum_{e \in T} \chi(e)$, thus identifying T with the multiset in $[n]$ in which the multiplicity of $i \in [n]$ is the number of sets in T containing i , counting with repetition. Also, we say, e.g. ‘a multiset T in J ’ to mean a multiset T of members of J .

4.1. Irreducibility. Now we make some important definitions. Given $b \in \mathbb{N}$, $u, v \in V(J)$ and multisets T in J and T' in M , we say that (T, T') is a *b -fold (u, v) -transferral in (J, M)* if

$$\chi(T) - \chi(T') = b(\chi(\{u\}) - \chi(\{v\})).$$

That is to say that every vertex of J appears equally many times in T as in T' , with the exception of u , which appears b times more in T than in T' , and v , which appears b times more in T' than in T . Note that if (T, T') is a b -fold (u, v) -transferral in (J, M) then we must have $|T| = |T'|$; we refer to this common size as the *size* of the transferral. We say that (J, M) is *(B, C) -irreducible* if for any $u, v \in V(J)$ there exist $b \leq B$ and $c \leq C$ such that (J, M) contains a b -fold (u, v) -transferral of size c . We also make the following partite version of this definition, in which we only require transferrals within parts. Given a partition \mathcal{P} of $V(J)$ into parts V_1, \dots, V_r , then we say that (J, M) is *(B, C) -irreducible with respect to \mathcal{P}* if for any $i \in [r]$ and any $u, v \in V_i$ there exist $b \leq B$ and $c \leq C$ such that (J, M) contains a b -fold (u, v) -transferral of size c . As mentioned in Section 2.6, irreducibility will allow us to reduce the cluster imbalances to a constant, and so find a matching in original system covering all but a constant number of vertices.

It is instructive to consider how irreducibility fails in the space barrier constructions (Construction 2.2). Consider for simplicity the 3-complex $J = J(S, 1)$ when $|S| = n/3$ (suppose that $3 \mid n$). By definition we have $|e \cap S| \geq 1$ for every $e \in J_3$. Then for any perfect matching M in J_3 , each $e' \in M$ has $|e' \cap S| = 1$. Thus for any multisets T, T' in J , M with $|T| = |T'|$, the vector $\mathbf{x} = \chi(T) - \chi(T')$ satisfies $\sum_{i \in S} x_i \geq 0$. It follows that there are no b -fold (u, v) -transferrals for any $b \in \mathbb{N}$, $v \in S$ and $u \notin S$. Conversely, we will

see in the next section that if there is no space barrier, then the minimum degree sequence assumption implies irreducibility (with respect to the partition in the partite case). In fact, it implies a geometric condition which is the fundamental property behind irreducibility. To formulate this, we define

$$X = X(J, M) = \{\chi(e) - \chi(e') : e \in J, e' \in M\}.$$

Also, if \mathcal{P} is a partition of V we define

$$\Pi_{\mathcal{P}} = \{\mathbf{x} \in \mathbb{Z}^n : \mathbf{x} \cdot \chi(U) = 0 \text{ for all } U \in \mathcal{P}\}.$$

The required geometric condition is that the convex hull of X should contain a ball about the origin within $\Pi_{\mathcal{P}}$. The following lemma shows that this implies irreducibility. (For now we gloss over the fact that B and C depend on n , which is not permissible in our proof strategy; this dependence will be removed by a random reduction in Lemma 5.4.)

Lemma 4.1. *Suppose that $1/B, 1/C \ll \delta, 1/n, 1/k$. Let V be a set of n vertices and let \mathcal{P} partition V . Suppose that (J, M) is a matched k -graph on V such that $X = X(J, M)$ satisfies $B(\mathbf{0}, \delta) \cap \Pi_{\mathcal{P}} \subseteq CH(X)$. Then (J, M) is (B, C) -irreducible with respect to \mathcal{P} .*

Proof. Let q be such that $1/B, 1/C \ll 1/q \ll \delta, 1/n, 1/k$. Fix any $U \in \mathcal{P}$ and $u, v \in U$. Let $t = \lceil 2/\delta \rceil$, and let $\mathbf{x} \in \mathbb{Q}^n$ have co-ordinates $x_u = 1/t$, $x_v = -1/t$ and $x_w = 0$ for $w \neq u, v$. Then $\|\mathbf{x}\| < \delta$ and $\mathbf{x} \in \Pi_{\mathcal{P}}$, so our hypothesis implies that $\mathbf{x} \in CH(X)$. Applying Proposition 3.2, we may choose $\lambda_1, \dots, \lambda_s \geq 0$ and $\mathbf{x}_1, \dots, \mathbf{x}_s \in X$ such that $s \leq n+1$, each λ_j is q -rational, $\sum_{j \in [s]} \lambda_j = 1$, and $\mathbf{x} = \sum_{j \in [s]} \lambda_j \mathbf{x}_j$. By definition of X , we can choose $e_j \in J$ and $e'_j \in M$ such that $\chi(e_j) - \chi(e'_j) = \mathbf{x}_j$ for $j \in [s]$. Now let the multiset T in J consist of $tq!\lambda_j$ copies of e_j for each $j \in [s]$, and similarly let the multiset T' of edges of M consist of $tq!\lambda_j$ copies of e'_j for each $j \in [s]$. Then

$$\chi(T) - \chi(T') = tq! \sum_{j \in [s]} \lambda_j \mathbf{x}_j = tq! \mathbf{x}.$$

So the paired multiset (T, T') is a $q!$ -fold (u, v) -transferral in (J, M) with size at most $tq! \leq C$. Since $U \in \mathcal{P}$ and $u, v \in U$ were arbitrary, we deduce that (J, M) is (B, C) -irreducible with respect to \mathcal{P} , as required. \square

4.2. Transferral digraphs. For perfect matchings, we need more precise balancing operations, namely simple transferrals, by which we mean 1-fold transferrals. We represent these using digraphs (directed graphs), for which we make the following standard definitions. A *digraph* D consists of a vertex set $V(D)$ and an edge set $E(D)$, where the edges are each ordered pairs of vertices; we allow loops (u, u) , and 2-cycles $\{(u, v), (v, u)\}$, but do not allow multiple copies of a given ordered pair. We think of an edge $(u, v) \in E(D)$ as being *directed* from u to v ; if $(u, v) \in E(D)$ then we say that v is an *outneighbour* of u and that u is an *inneighbour* of v . For any vertex $v \in V(D)$, the *outneighbourhood* $N^+(v)$ is the set of outneighbours of v , and the *inneighbourhood* $N^-(v)$ is the set of inneighbours of v . We also define the *outdegree* $d^+(v) = |N^+(v)|$ and the *indegree* $d^-(v) = |N^-(v)|$, and the *minimum outdegree* $\delta^+(D)$ of D , which is the minimum of $d^+(v)$ over all $v \in D$.

We represent the simple transferrals in (J, M) by the ℓ -*transferral digraph* of (J, M) , denoted $D_{\ell}(J, M)$, where $\ell \in \mathbb{N}$. This is defined to be the digraph on $V(J)$ in which there is an edge from u to v if and only if (J, M) contains a simple (u, v) -transferral of size at most ℓ . For future reference we record here some basic properties of transferral digraphs:

- (i) $D_\ell(J, M)$ contains all loops (u, u) , as a (u, u) -transferral is trivially achieved by the empty multisets.
- (ii) If $\ell \leq \ell'$ then $D_\ell(J, M)$ is a subgraph of $D_{\ell'}(J, M)$.
- (iii) If $(u, v) \in D_\ell(J, M)$ and $(v, w) \in D_{\ell'}(J, M)$ then $(v, w) \in D_{\ell+\ell'}(J, M)$. Indeed, if (S, S') is a simple (u, v) -transferral of size at most ℓ and (T, T') is a simple (v, w) -transferral of size at most ℓ' , then $(S + T, S' + T')$ is a simple (u, w) -transferral of size at most $\ell + \ell'$.

In the last property we used the notation $A + B$ for the multiset union of multisets A and B . We will also use the notation $\sum_{i \in [t]} A_i$ for the multiset union of multisets A_i , $i \in [t]$, and the notation pA for the multiset union of p copies of A .

Our goal in the remainder of this section is to describe the conditions under which, given a matched k -graph (J, M) on V and a partition \mathcal{P} of V , there is some transferral digraph that is complete on each part of \mathcal{P} , i.e. there is some ℓ so that $(u, v) \in D_\ell(J, M)$ for every $u, v \in U \in \mathcal{P}$. This is a convenient context that in fact gives more general results than the two cases that are needed for our main theorems, namely (i) when \mathcal{P} consists of a single part V , and (ii) when \mathcal{P} consists of at least k parts and (J, M) is \mathcal{P} -partite. In this subsection, we prove a structure theorem for general digraphs under a minimum outdegree condition. This introduces a partition \mathcal{P}' that refines \mathcal{P} , and we show that irreducibility implies that there is some C' for which $D_{C'}(J, M)$ is complete on the parts of \mathcal{P}' . In the final subsection we show that if there is no divisibility barrier then there is some C for which $D_C(J, M)$ is complete on the parts of \mathcal{P} , as required.

We require the following lemma, which states that in a digraph with linear minimum outdegree, one can choose a constant number of vertices so that from any fixed vertex there is a short directed path to one of these chosen vertices.

Lemma 4.2. *Suppose that $1/\ell \ll \alpha$, and let D be a digraph on n vertices. Then we may choose vertices $v_1, \dots, v_t \in V(D)$ and sets $V_1, \dots, V_t \subseteq V(D)$ such that*

- (a) *for any j and any $u \in V_j$ there exists a directed path from u to v_j of length at most ℓ ,*
- (b) *the sets V_1, \dots, V_t partition $V(D)$, and*
- (c) *$|V_j| \geq \delta^+(D) - \alpha n$ for each j .*

Proof. Let γ' satisfy $1/(\ell - 1) \ll \gamma' \ll \alpha$. Clearly we may assume that $\delta^+(D) \geq \alpha n$. For any $S \subseteq V(D)$, we let $L(S)$ be the number of edges of D that *leave* S , i.e. are of the form (u, v) with $u \in S$ and $v \in \overline{S} = V(D) \setminus S$. We require the following claim.

Claim 4.3. *Let $1/(\ell - 1) \leq \gamma \leq \gamma'$, and let $S \subseteq V(D)$ be such that $|S| \geq \alpha n$ and $L(S) \leq \gamma n^2$. Then we may choose $v \in S$ and $S' \subseteq S$ such that $|S'| > \delta^+(D) - \alpha n$, $L(S \setminus S') \leq 2\gamma n^2$, and for any $u \in S'$ there is a path from u to v in D of length at most $\ell - 1$.*

To prove the claim, we start by showing that there must be some $v \in S$ with $d_S^-(v) := |N^-(v) \cap S| \geq \delta^+(D) - \alpha n$. To see this, we note that $\sum_{v \in S} d_S^-(v) \geq \sum_{v \in S} d^+(v) - L(S) \geq |S|\delta^+(D) - \gamma n^2$. Thus by averaging we can choose $v \in S$ with $d_S^-(v) \geq \delta^+(D) - \gamma n^2/|S| \geq \delta^+(D) - \alpha n$. Now consider the ‘iterated inneighbourhood’ N_j^- of v for $j \geq 1$, defined as the set of vertices $u \in S$ such that there exists a path from u to v in D of length at most j . Note that $N_1^- \subseteq N_2^- \subseteq \dots$, so we can choose some $j \leq 1/\gamma$ for which $|N_{j+1}^-| \leq |N_j^-| + \gamma n$. Now we set $S' = N_j^-$. By construction this satisfies the required property of paths (since $j \leq 1/\gamma \leq \ell - 1$). We also have $|S'| \geq |N_1^-| \geq \delta^+(D) - \alpha n$, so it remains to estimate

$L(S \setminus S')$. Any edge leaving $S \setminus S'$ is either an edge leaving S or an edge from $S \setminus S'$ to S' . By assumption there are at most γn^2 edges of the first type. Also, for any edge (u, v) of the second type we have $u \in N_{j+1}^- \setminus N_j^-$, so there are at most $|N_{j+1}^- \setminus N_j^-|n \leq \gamma n^2$ such edges. This completes the proof of Claim 4.3.

Returning to the proof of the lemma, we apply the claim to $S = V(D)$ with $\gamma = 1/(\ell - 1)$, obtaining $v \in S$ and $S' \subseteq S$ with the stated properties. We set $v_1 = v$ and $V_1 = S'$. Then we repeatedly apply the claim for each $j \geq 1$ to $S = V(D) \setminus (V_1 \cup \dots \cup V_j)$ with $\gamma = 2^j/(\ell - 1)$; in each application we obtain $v \in S$ and $S' \subseteq S$ for the current set S and we set $v_{j+1} = v$ and $V_{j+1} = S'$. Since $\delta^+(D) \geq \alpha n$, we can repeat this process until we reach some $t \geq 1$ for which $S = V(D) \setminus (V_1 \cup \dots \cup V_t)$ has $|S| < \delta^+(D)$. Then any $u \in S$ must have an outneighbour $u' \in V(D) \setminus S$, i.e. $u' \in V_j$ for some $j \in [t]$. By choice of V_j there is a path of length at most $\ell - 1$ from u' to v_j , so we have a path of length at most ℓ from u to v_j . For each such u , choose any such j arbitrarily, and add u to V_j . Then the vertices v_1, \dots, v_t and the new sets V_1, \dots, V_t satisfy the properties required for the lemma. \square

Now we define the structure we will find in our digraphs. Suppose D is a digraph. We say that $S \subseteq V(D)$ is a *receiving set* in D if (u, v) is an edge of D for every $u \in V(D)$ and $v \in S$. Now let \mathcal{P} be a partition of $V(D)$ into parts V_1, \dots, V_r . We say that $\mathcal{S} = (S_1, \dots, S_r)$ with $S_1 \subseteq V_1, \dots, S_r \subseteq V_r$ is a *receiving \mathcal{P} -tuple* if each S_j is a receiving set in $D[V_j]$. If such an \mathcal{S} exists, we say that \mathcal{P} is a *receiving partition* of D . The ℓ th power of D is the digraph D^ℓ with vertex set $V(D)$ where $(u, v) \in D^\ell$ if and only if there exists a path from u to v in D of length at most ℓ . Now we can state our structural result, which is that some power of D admits a receiving partition.

Corollary 4.4. *Suppose that $1/\ell \ll \alpha$, and let D be a digraph on n vertices. Then D^ℓ admits a receiving partition \mathcal{P} in which each part has size at least $\delta^+(D) - \alpha n$.*

Proof. Apply Lemma 4.2 to obtain sets $V_1, \dots, V_t \subseteq V(D)$ and $v_1, \dots, v_t \in V(D)$ for some t . Let \mathcal{P} be the partition of $V(D)$ into parts V_1, \dots, V_t . So each part of \mathcal{P} has size at least $\delta^+(D) - \alpha n$. Now, for any $j \in [t]$ and any $u \in V_j$ there is a path in D from u to v_j of length at most ℓ , and therefore an edge in D^ℓ from u to v_j . Then $\mathcal{S} := (\{v_1\}, \dots, \{v_t\})$ is a receiving \mathcal{P} -tuple in D^ℓ , as required. \square

Now we show that in combination with irreducibility, a receiving partition \mathcal{P} has the property that some transferral digraph is complete on every part of \mathcal{P} . We need the following lemma.

Lemma 4.5. *Let (J, M) be a matched k -graph, $b, c, c' \in \mathbb{N}$, and vertices $u, v \in V(J)$ be such that (J, M) contains a b -fold (u, v) -transferral of size c , and (J, M) contains a simple (v, u) -transferral of size c' . Then (J, M) contains a simple (u, v) -transferral of size $(b - 1)c' + c$.*

Proof. By definition of transferrals, there are multisets T_1, T_2 in J and T'_1, T'_2 in M such that $|T_1| = |T'_1| = c$, $|T_2| = |T'_2| = c'$,

$$\chi(T_1) - \chi(T'_1) = b(\chi(\{u\}) - \chi(\{v\})), \text{ and } \chi(T_2) - \chi(T'_2) = \chi(\{v\}) - \chi(\{u\}).$$

Let $T = T_1 + (b - 1)T_2$ and $T' = T'_1 + (b - 1)T'_2$. Then $|T| = |T'| = (b - 1)c' + c$, and

$$\chi(T) - \chi(T') = \chi(T_1) - \chi(T'_1) + (b - 1)(\chi(T_2) - \chi(T'_2)) = \chi(\{u\}) - \chi(\{v\}),$$

so (T, T') is the desired transferral. \square

Finally, we show the required completeness property of the transferral digraph.

Lemma 4.6. *Suppose that $1/\ell' \ll 1/B, 1/C, 1/\ell \ll \alpha$. Let \mathcal{P}' be a partition of a set V of size n and (J, M) be a matched k -graph on V which is (B, C) -irreducible with respect to \mathcal{P}' . Suppose also that $\delta^+(D_\ell(J, M)[U]) \geq \gamma n$ for every $U \in \mathcal{P}'$. Then $D_{\ell^2}(J, M)$ admits a partition \mathcal{P} refining \mathcal{P}' in which every part has size at least $(\gamma - \alpha)n$, and $D_{\ell'}(J, M)[U]$ is complete for every $U \in \mathcal{P}$.*

Proof. Applying Corollary 4.4 to $D_\ell(J, M)[U]$ for each $U \in \mathcal{P}'$, we see that $D_\ell(J, M)^\ell$ admits a receiving partition \mathcal{P} refining \mathcal{P}' in which every part has size at least $(\gamma - \alpha)n$. We have $D_\ell(J, M)^\ell \subseteq D_{\ell^2}(J, M)$ by property (iii) of transferral digraphs, so \mathcal{P} is also a receiving partition for $D_{\ell^2}(J, M)$. Now suppose $U \in \mathcal{P}$ and $S \subseteq U$ is the receiving set for $D_{\ell^2}(J, M)[U]$. Consider any $u, v \in U$. We need to show that there is a simple (u, v) -transferral of size at most ℓ' . Fix any $w \in S$. By choice of \mathcal{P} , we have a simple (u, w) -transferral and a simple (v, w) -transferral each of size at most ℓ^2 . By irreducibility, we also have a b -fold (w, v) -transferral of size c for some $b \leq B$ and $c \leq C$. Applying Lemma 4.5, we have a simple (w, v) -transferral of size $(b - 1)\ell^2 + c$. Combining this with the simple (u, w) -transferral, we obtain a simple (u, v) -transferral of size at most $b\ell^2 + c \leq \ell'$. \square

4.3. Completion of the transferral digraph. In this final subsection, we show that if in addition to the previous assumptions there is no divisibility barrier, then we can strengthen the previous structure to obtain a transferral digraph that is complete on every part of the original partition. We start with a useful consequence of irreducibility in the following lemma, which allows us to extend a pair of multisets in J and M to a pair of multisets that cover the same vertices (including multiplicities). This corresponds to the geometric intuition that we can use a ball about the origin to counterbalance any given vector. We need to impose the following property on (J, M) that will be automatically satisfied in our applications: we say that (J, M) is \mathcal{P} -proper if for any $e \in J$ there is $e' \in M$ with $\mathbf{i}_\mathcal{P}(e') = \mathbf{i}_\mathcal{P}(e)$.

Lemma 4.7. *Suppose that $1/C' \ll 1/B, 1/C, 1/k$, let \mathcal{P} be a partition of a set of vertices V , and let (J, M) be a matched k -graph on V which is \mathcal{P} -proper and (B, C) -irreducible with respect to \mathcal{P} . Then for any multisets A in J and A' in M with $|A|, |A'| \leq C$ there exist multisets T in J and T' in M such that $A \subseteq T$, $A' \subseteq T'$, $|T| = |T'| \leq C'$ and $\chi(T) = \chi(T')$.*

Proof. First we find multisets S in J and S' in M such that $A \subseteq S$, $A' \subseteq S'$ and $\sum_{e \in S} \mathbf{i}_\mathcal{P}(e) = \sum_{e' \in S'} \mathbf{i}_\mathcal{P}(e')$. To do this, we let $S = A + A'$ and $S' = A^* + A'$, where A^* is formed by taking some $e' \in M$ with $\mathbf{i}_\mathcal{P}(e') = \mathbf{i}_\mathcal{P}(e)$ for every $e \in A$ (which exists because (J, M) is \mathcal{P} -proper). Since $\sum_{e \in S} \mathbf{i}_\mathcal{P}(e) = \sum_{e' \in S'} \mathbf{i}_\mathcal{P}(e')$ we may label the vertices of S and S' as $u_1, \dots, u_{k|S|}$ and $v_1, \dots, v_{k|S|}$ respectively so that u_i and v_i lie in the same part of \mathcal{P} for each $i \in [k|S|]$. By irreducibility, for each $i \in [k|S|]$ there is a b -fold (u_i, v_i) -transferral in (J, M) of size c for some $b \leq B$ and $c \leq C$. Combining $B!/b$ copies of this transferral, we obtain a $B!$ -fold (u_i, v_i) -transferral (T_i, T'_i) of size at most $CB!$. Let $T = B!S + \sum_{i \in [k|S|]} T_i$

and $T' = B!S' + \sum_{i \in [k|S|]} T'_i$. Then $|T| = |T'| \leq CB! + 2Ck \cdot CB! \leq C'$. Also,

$$\begin{aligned} \chi(T) - \chi(T') &= B!(\chi(S) - \chi(S')) + \sum_{i \in [k|S|]} \chi(T_i) - \chi(T'_i) \\ &= B! \sum_{i \in [k|S|]} (\chi(\{u_i\}) - \chi(\{v_i\}) - \chi(\{u_i\}) + \chi(\{v_i\})) = 0, \end{aligned}$$

which completes the proof. \square

Next we need a simple proposition that allows us to efficiently represent vectors in a lattice, in that we have bounds on the number of terms and size of the coefficients in the representation.

Proposition 4.8. *Suppose that $1/s \ll 1/k, 1/d$. Let $X \subseteq \mathbb{Z}^d \cap B^d(\mathbf{0}, 2k)$, and let L_X be the sublattice of \mathbb{Z}^d generated by X . Then for any vector $\mathbf{x} \in L_X \cap B^d(\mathbf{0}, 2k)$ we may choose multisets S_1 and S_2 of elements of X such that $|S_1|, |S_2| \leq s$ and $\sum_{\mathbf{v} \in S_1} \mathbf{v} - \sum_{\mathbf{v} \in S_2} \mathbf{v} = \mathbf{x}$.*

Proof. The number of pairs (X, \mathbf{x}) as in the statement of the proposition depends only on k and d . Furthermore, for any such pair (X, \mathbf{x}) we may choose multisets S_1 and S_2 of elements of X such that $\sum_{\mathbf{v} \in S_1} \mathbf{v} - \sum_{\mathbf{v} \in S_2} \mathbf{v} = \mathbf{x}$. Let $s_{X, \mathbf{x}}$ be minimal such that we may do this with $|S_1|, |S_2| \leq s_{X, \mathbf{x}}$. Since $1/s \ll 1/k, 1/d$, we may assume that $s \geq \max_{(X, \mathbf{x})} s_{X, \mathbf{x}}$. Thus for any X and \mathbf{x} we may express \mathbf{x} in the required manner. \square

After these preparations, we are ready to prove the main lemma of this section. Suppose that \mathcal{P}' is a partition of a set V and \mathcal{P} is a partition into d parts refining \mathcal{P}' . Given a k -graph G on V , we define the *edge lattice* $L_{\mathcal{P}}(G) \subseteq \mathbb{Z}^d$ to be the lattice generated by all vectors $\mathbf{i}_{\mathcal{P}}(e)$ with $e \in G$. (Note that this definition is similar to that of the robust edge lattices defined earlier; indeed the purpose of robustness is for the same edge lattice to be inherited by the reduced graph.) Recall that a lattice $L \subseteq \mathbb{Z}^d$ is complete with respect to \mathcal{P}' if $L_{\mathcal{P}\mathcal{P}'} \subseteq L \cap \Pi^d$, where $L_{\mathcal{P}\mathcal{P}'} \subseteq \mathbb{Z}^d \cap \Pi^d$ is the lattice generated by all differences of basis vectors $\mathbf{u}_i - \mathbf{u}_j$ for which V_i, V_j are contained in the same part of \mathcal{P}' and $\Pi^d = \{\mathbf{x} \in \mathbb{R}^d : \sum_{i \in [d]} x_i = 0\}$.

Lemma 4.9. *Suppose that $1/C' \ll 1/B, 1/C, 1/d, 1/k$. Let \mathcal{P}' be a partition of a set of vertices V , and let (J, M) be a matched k -graph on V which is \mathcal{P}' -proper and (B, C) -irreducible with respect to \mathcal{P}' . Suppose that \mathcal{P} is a partition of V into d parts V_1, \dots, V_d which refines \mathcal{P}' such that $D_C(J, M)[V_j]$ is complete for each $j \in [d]$ and $L_{\mathcal{P}}(J)$ is complete with respect to \mathcal{P}' . Then (J, M) is $(1, C')$ -irreducible with respect to \mathcal{P}' , i.e. $D_{C'}(J, M)[U]$ is complete for each $U \in \mathcal{P}'$.*

Proof. Introduce new constants with $1/C' \ll 1/C_2 \ll 1/C_1 \ll 1/C_0 \ll 1/B, 1/C, 1/d, 1/k$. Fix any $i, j \in [d]$ with $i \neq j$ such that V_i and V_j are both subsets of the same part of \mathcal{P}' (if no such i, j exist then there is nothing to prove). Then $\mathbf{x} := \mathbf{u}_i - \mathbf{u}_j \in L_{\mathcal{P}\mathcal{P}'} \subseteq L_{\mathcal{P}}(J)$ by assumption. By Proposition 4.8 we may therefore choose multisets S and T in J such that $\mathbf{i}_{\mathcal{P}}(S) - \mathbf{i}_{\mathcal{P}}(T) = \mathbf{x}$ and $|S|, |T| \leq C_0$. Note that $\sum_{j \in [d]} \mathbf{i}_{\mathcal{P}}(S)_j - \sum_{j \in [d]} \mathbf{i}_{\mathcal{P}}(T)_j = \sum_{j \in [d]} x_j = 0$, $\sum_{j \in [d]} \mathbf{i}_{\mathcal{P}}(S)_j = k|S|$ and $\sum_{j \in [d]} \mathbf{i}_{\mathcal{P}}(T)_j = k|T|$, so S and T are of equal size. So we may label the vertices of S as u_1, \dots, u_r and T as v_1, \dots, v_r so that $u_1 \in V_i$, $v_1 \in V_j$, and u_ℓ and v_ℓ lie in the same part of \mathcal{P} for every $2 \leq \ell \leq r$. Since $D_C(J, M)[V_\ell]$ is complete for each

$\ell \in [r]$, for each $2 \leq \ell \leq r$ we may choose (T_ℓ, T'_ℓ) to be a simple (u_ℓ, v_ℓ) -transferral of size at most C . Then

$$\chi(S) - \chi(T) = \sum_{\ell \in [r]} \chi(\{u_\ell\}) - \chi(\{v_\ell\}) = \chi(\{u_1\}) - \chi(\{v_1\}) + \sum_{2 \leq \ell \leq r} \chi(T_\ell) - \chi(T'_\ell).$$

Now, by Lemma 4.7 there exist multisets A_1 in J and A'_1 in M such that $|A_1| = |A'_1| \leq C_1$, $S \subseteq A_1$ and $\chi(A_1) = \chi(A'_1)$. Let $A_2 = A_1 - S + T$ be formed by deleting S from A_1 and replacing it by T . Then $\chi(A_2) - \chi(A'_1) = \chi(A_1) - \chi(S) + \chi(T) - \chi(A'_1) = \chi(T) - \chi(S)$. Finally, let $A_3 = A_2 + \sum_{\ell=2}^r T_\ell$ and $A'_3 = A'_1 + \sum_{\ell=2}^r T'_\ell$. Then

$$\begin{aligned} \chi(A_3) - \chi(A'_3) &= \chi(A_2) - \chi(A'_1) + \sum_{2 \leq \ell \leq r} \chi(T_\ell) - \chi(T'_\ell) \\ &= \chi(\{v_1\}) - \chi(\{u_1\}). \end{aligned}$$

So (A_3, A'_3) is a simple (v_1, u_1) -transferral of size at most C_2 . Since $D_C(J, M)[V_i]$ and $D_C(J, M)[V_j]$ are each complete, we deduce that there exists a simple (v, u) -transferral of size at most $C_2 + 2C \leq C'$ for each $u \in V_i$ and $v \in V_j$. This holds for any i, j such that V_i and V_j are both subsets of the same part U of \mathcal{P}' , so $D_{C'}(J, M)[U]$ is complete for each $U \in \mathcal{P}'$. \square

Combining Lemmas 4.9 and 4.6 we obtain the following result.

Lemma 4.10. *Suppose that $1/\ell' \ll 1/B, 1/C, 1/\ell \ll \alpha \ll \gamma$. Let \mathcal{P}' be a partition of a set V of size n and (J, M) be a matched k -graph on V that is \mathcal{P}' -proper and (B, C) -irreducible with respect to \mathcal{P}' . Suppose that $\delta^+(D_\ell(J, M)[U]) \geq \gamma n$ for any $U \in \mathcal{P}'$, and $L_{\mathcal{P}}(J)$ is complete with respect to \mathcal{P}' for any partition \mathcal{P} of $V(J)$ into d parts of size at least $(\gamma - \alpha)n$ that refines \mathcal{P}' . Then $D_{\ell'}(J, M)[U]$ is complete for every $U \in \mathcal{P}'$.*

5. TRANSFERRALS VIA THE MINIMUM DEGREE SEQUENCE

In the previous section we demonstrated how irreducibility and completeness of the edge lattice imply the existence of all possible simple transferrals. In this section we will obtain the same result in the minimum degree sequence settings of our main theorems. In fact we will work in the following setting that simultaneously generalises the partite and non-partite settings.

Let J be a k -system on V and \mathcal{P} be a partition of V into sets V_1, \dots, V_r . An *allocation function* is a function $f : [k] \rightarrow [r]$. Intuitively an allocation function should be viewed as a rule for forming an edge of J by choosing vertices one by one; the i th vertex should be chosen from part $V_{f(i)}$ of \mathcal{P} . This naturally leads to the notion of the *minimum f -degree sequence* $\delta^f(J) = (\delta_0^f(J), \dots, \delta_{k-1}^f(J))$, where $\delta_j^f(J)$ is the largest m such that for any $\{v_1, \dots, v_j\} \in J$ with $v_i \in V_{f(i)}$ for $i \in [j]$ there are at least m vertices $v_{j+1} \in V_{f(j+1)}$ such that $\{v_1, \dots, v_{j+1}\} \in J$. So $\delta_{i-1}^f(J)$ is a lower bound on the number of choices for the i th vertex in the process of forming an edge of J described above. For any allocation function f , the index vector $\mathbf{i}(f)$ of f is the vector in \mathbb{Z}^r whose j th coordinate is the cardinality of the preimage of j under f . So any edge $e \in J_k$ formed by the described process will have $\mathbf{i}(e) = \mathbf{i}(f)$.

Let I be a multiset of index vectors with respect to \mathcal{P} such that $\sum_{j \in [r]} i_j = k$ for each $\mathbf{i} = (i_1, \dots, i_r)$ in I . Then we may form a multiset F of allocation functions $f : [k] \rightarrow [r]$

as follows. For each $\mathbf{i} \in I$ (with repetition) choose an allocation function f with $\mathbf{i}(f) = \mathbf{i}$, and include in F each of the $k!$ permutations f^σ for $\sigma \in \text{Sym}_k$ (again including repetitions), where $f^\sigma(i) = f(\sigma(i))$ for $i \in [k]$. Note that the multiset F so obtained does not depend on the choices of allocation function f . Also observe that $|F| = k!|I|$, and for any $f \in F$ with $\mathbf{i}(f) = \mathbf{i} \in I$, the multiplicity of f in F is $m_{\mathbf{i}} \prod_{j \in [r]} i_j!$, where $m_{\mathbf{i}}$ is the multiplicity of \mathbf{i} in I . If F can be obtained in this manner, then we say that F is an *allocation*, and we write $I(F)$ for the multiset I from which F was obtained. We say that an allocation F is (k, r) -uniform if for every $i \in [k]$ and $j \in [r]$ there are $|F|/r$ functions $f \in F$ with $f(i) = j$. We also say that F is *connected* if there is a connected graph G_F on $[r]$ such that for every $jj' \in E(G_F)$ and $i, i' \in [k]$ with $i \neq i'$ there is $f \in F$ with $f(i) = j$ and $f(i') = j'$.

All the allocations F considered in this paper will have the property that $I(F)$ is a set (as opposed to a multiset). In this case we have $I(F) = \{\mathbf{i}(f) : f \in F\}$, and so $|I| \leq r^k$ and $|F| \leq k!|I| \leq k!r^k$. However, we allow the possibility that $I(F)$ is a multiset as this may be useful for future applications. In this case, we shall usually bound $|F|$ by a function $D_F(r, k)$. Also, the reader should take care with any statement regarding (e.g.) $\mathbf{i} \in I(F)$ to interpret the statement with multiplicity. We slightly abuse the notation along these lines, writing (e.g.) “the number of edges of index \mathbf{i} is constant over all $\mathbf{i} \in I(F)$ ” to mean that the number of edges of index \mathbf{i} is a constant multiple of the multiplicity of \mathbf{i} in $I(F)$.

For an allocation F , a k -system J on V is \mathcal{PF} -partite if for any $j \in [k]$ and $e \in J_j$ there is some $f \in F$ so that $e = \{v_1, \dots, v_j\}$ with $v_i \in V_{f(i)}$ for $i \in [j]$ (so every edge of J can be constructed through the process above for some $f \in F$). The *minimum F -degree sequence* of J is then defined to be $\delta^F(J) = (\delta_0^F(J), \dots, \delta_{k-1}^F(J))$, where $\delta_j^F(J) = \min_{f \in F} \delta_j^f(J)$. We note two special cases of this definition that recover our earlier settings. In the case $r = 1$ (the non-partite setting), there is a unique function $f : [k] \rightarrow [1]$. Let F consist of $k!$ copies of f ; then J is \mathcal{PF} -partite and $\delta^F(J) = \delta(J)$ is the (usual) minimum degree sequence. In the case when $r \geq k$ and J is \mathcal{P} -partite we can instead choose F to consist of all injective functions from $[k]$ to $[r]$, and then being \mathcal{P} -partite is equivalent to being \mathcal{PF} -partite and $\delta^F(J) = \delta^*(J)$ is the partite minimum degree sequence. Note that in both cases F is (k, r) -uniform and connected.

Recall that we established a geometric criterion for irreducibility in Lemma 4.1. Our first lemma states that if there is no space barrier then this criterion holds under our generalised minimum degree assumption. Note we have slightly altered the formulation of the space barrier from that used earlier, so that we can state the lemma in the more general context of k -systems; it is not hard to demonstrate that in the case of k -complexes it is an essentially equivalent condition (up to changing the value of β). We say that (J, M) is a *matched k -system* if J is a k -system and M is a perfect matching in J_k .

Lemma 5.1. *Suppose that $\delta \ll 1/n \ll \alpha \ll \beta, 1/D_F, 1/r, 1/k$, and $k \mid n$. Let \mathcal{P}' be a partition of a set V into sets V_1, \dots, V_r each of size n , F be a (k, r) -uniform connected allocation with $|F| \leq D_F$, and (J, M) be a $\mathcal{P}'F$ -partite matched k -system on V such that*

$$\delta^F(J) \geq \left(n, \left(\frac{k-1}{k} - \alpha \right) n, \left(\frac{k-2}{k} - \alpha \right) n, \dots, \left(\frac{1}{k} - \alpha \right) n \right).$$

Suppose that for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ such that $|S_i| = pn/k$ for each $i \in [r]$ we have $|J_{p+1}[S]| \geq \beta n^{p+1}$, where $S = S_1 \cup \dots \cup S_r$. Then

$$B^{rn}(\mathbf{0}, \delta) \cap \Pi_{\mathcal{P}'} \subseteq CH(X(J, M)).$$

Proof. We start by applying Lemma 3.5 to $X = X(J, M) \subseteq \mathbb{Z}^{rn} \cap B^{rn}(\mathbf{0}, 2k)$. This gives $\Pi_0^X \cap B^{rn}(\mathbf{0}, \delta) \subseteq CH(X)$, where $F_0^X = CH(X) \cap \Pi_0^X$ is the minimum face of $CH(X)$ containing $\mathbf{0}$. We can write $\Pi_0^X = \{\mathbf{x} \in \mathbb{Z}^{rn} : \mathbf{a} \cdot \mathbf{x} = 0 \text{ for all } \mathbf{a} \in A\}$ for some $A \subseteq \mathbb{Z}^{rn}$ such that $\mathbf{a} \cdot \mathbf{x} \leq 0$ for every $\mathbf{x} \in X$ and $\mathbf{a} \in A$. To prove that $B^{rn}(\mathbf{0}, \delta) \cap \Pi_{\mathcal{P}'} \subseteq CH(X)$, it suffices to show that A is contained in the subspace generated by $\{\chi(V_i) : i \in [r]\}$. Suppose for a contradiction that some $\mathbf{a} \in A$ is not in $\langle \{\chi(V_i) : i \in [r]\} \rangle$. Then there is some $i' \in [r]$ such that \mathbf{a} is not constant on $V_{i'}$.

We label V so that for each $i \in [r]$, the vertices of V_i are labelled $v_{i,1}, \dots, v_{i,n}$, the corresponding coordinates of \mathbf{a} are $a_{i,1}, \dots, a_{i,n}$, and

$$a_{i,1} \leq a_{i,2} \leq \dots \leq a_{i,n}.$$

It is convenient to assume that we have a strict inequality $\delta_i^F(J) > \left(\frac{k-i}{k} - \alpha\right)n$ for $i \in [k-1]$. This can be achieved by replacing α with a slightly smaller value; we can also assume that $\alpha n \in \mathbb{N}$. Next we partition each V_i into k parts $V_{i,1}, \dots, V_{i,k}$ as follows. For $i \in [r]$ we let

$$V_{i,j} = \begin{cases} \{v_{i,\ell} : 0 < \ell \leq n/k - \alpha n\} & \text{if } j = 1, \\ \{v_{i,\ell} : (j-1)n/k - \alpha n < \ell \leq jn/k - \alpha n\} & \text{if } 2 \leq j \leq k-1, \\ \{v_{i,\ell} : (k-1)n/k - \alpha n < \ell \leq n\} & \text{if } j = k. \end{cases}$$

Then for each $i \in [r]$ we have

$$(1) \quad |V_{i,j}| = \begin{cases} \frac{n}{k} - \alpha n & \text{if } j = 1, \\ \frac{n}{k} & \text{if } 2 \leq j \leq k-1, \\ \frac{n}{k} + \alpha n & \text{if } j = k. \end{cases}$$

We use these partitions to define a simpler vector \mathbf{a}' that will be a useful approximation to \mathbf{a} . For $i \in [r]$ and $j \in [n]$ we let $p(j) \in [k]$ be such that $v_{i,j} \in V_{i,p(j)}$. For $i \in [r]$ and $\ell \in [k]$ we let $u_{i,\ell} = \max\{a_{i,j} : p(j) = \ell\}$. Then we define $\mathbf{a}' \in \mathbb{Z}^{rn}$ by

$$\mathbf{a}' := (a'_{i,j} : i \in [r], j \in [n]), \text{ where } a'_{i,j} = u_{i,p(j)}.$$

Note that $a'_{i,j} \geq a_{i,j}$ for every $i \in [r]$ and $j \in [n]$. We can assume without loss of generality that $u_{i,k} - u_{i,1}$ is maximised over $i \in [r]$ when $i = 1$; then we define

$$d = u_{1,k} - u_{1,1}, \quad U = \frac{1}{r} \sum_{i \in [r], j \in [k]} u_{i,j}, \text{ and } U_f = \sum_{j \in [k]} u_{f(j),j} \text{ for } f \in F.$$

Note that uniformity of F implies that U is the average of U_f over $f \in F$. We will prove the following claim.

Claim 5.2. *We have the following properties.*

- (i) *For any $e' \in M$ and $f \in F$ we have $\mathbf{a}' \cdot \chi(e') \geq \mathbf{a} \cdot \chi(e') \geq U_f$ and $\mathbf{a} \cdot \chi(e') \geq U$,*
- (ii) *$d > 0$ and $\sum_{e' \in M} (\mathbf{a}' \cdot \chi(e') - U) \leq rd\alpha n$,*
- (iii) *There is some $e' \in M$ such that $\mathbf{a} \cdot \chi(e') \leq U_f + D_F k d \alpha$ for every $f \in F$,*
- (iv) *For any $i, i' \in [r]$ and $j, j' \in [k]$ we have $|(u_{i,j} - u_{i,j'}) - (u_{i',j} - u_{i',j'})| \leq D_F r k d \alpha$.*

To see property (i), note that the first inequality follows from $a'_{i,j} \geq a_{i,j}$. For the second, we use the minimum f -degree sequence of J to greedily form an edge $e = \{v_{f(k),\ell_k}, \dots, v_{f(1),\ell_1}\} \in J_k$, starting with $\ell_k = n$, then choosing $\ell_j > jn/k - \alpha n$ for each $j = k-1, k-2, \dots, 1$. Then $p(\ell_j) \geq j+1$ for $j \in [k-1]$, so $\mathbf{a} \cdot \chi(e) \geq U_f$. Now the inequality follows from the

definition of A , which gives $\mathbf{a} \cdot (\chi(e') - \chi(e)) \geq 0$. We noted above that U can be expressed as an average of some U_f 's, so $\mathbf{a} \cdot \chi(e') \geq U$. This proves property (i).

To see that $d > 0$, suppose for a contradiction that $u_{i,k} = u_{i,1}$ for $i \in [r]$. Recall that there is some $i' \in [r]$ such that \mathbf{a} is not constant on $V_{i'}$. Let $e' \in M$ be the edge of M containing $v_{i',1}$. Since (J, M) is $\mathcal{P}'F$ -partite we can write $e' = \{v_{f(i),n_i}\}_{i \in [k]}$ for some $f \in F$ and $n_i \in [n]$ for $i \in [k]$, where without loss of generality $v_{f(1),n_1} = v_{i',1}$. Then $\mathbf{a} \cdot \chi(e') \leq a_{i',1} + \sum_{i=2}^k a_{f(i),n_i} < \sum_{i=1}^k a_{f(i),n_i} = U_f$. However, this contradicts property (i), so $d > 0$. Next note that $\sum_{e' \in M} \chi(e') = \mathbf{1}$ and $|M| = rn/k$, so by (1) and the definition of d we have

$$\begin{aligned} \sum_{e' \in M} (\mathbf{a}' \cdot \chi(e') - U) &= \mathbf{a}' \cdot \mathbf{1} - |M|U = \sum_{i \in [r], j \in [k]} (|V_i| - n/k)u_{i,j} \\ &= \sum_{i \in [r]} (u_{i,k} - u_{i,1})\alpha n \leq rd\alpha n, \end{aligned}$$

so we have proved property (ii).

Now we may choose some $e' \in M$ with $0 \leq \mathbf{a}' \cdot \chi(e') - U \leq rd\alpha n/|M| = kd\alpha$. Since $U = |F|^{-1} \sum_{f \in F} U_f$, $\mathbf{a}' \cdot \chi(e') \geq \mathbf{a} \cdot \chi(e') \geq U_f$ for any $f \in F$ and $|F| \leq D_F$, we have $0 \leq \mathbf{a}' \cdot \chi(e') - U_f \leq D_F kd\alpha$; this implies (iii), using $\mathbf{a}' \cdot \chi(e') \geq \mathbf{a} \cdot \chi(e')$. It also implies $|U_f - U_{f'}| \leq D_F kd\alpha$ for any $f, f' \in F$. Now, by definition of G_F and invariance of F under Sym_k , for any $i, i' \in [r]$ and $j, j' \in [k]$ such that $ii' \in G_F$, we can choose $f \in F$ with $f(j) = i$ and $f(j') = i'$, and let $f' \in F$ be obtained from f by transposing the values on j and j' . Then $|U_f - U_{f'}| = |(u_{i,j} - u_{i,j'}) - (u_{i',j} - u_{i',j'})| \leq D_F kd\alpha$. Since G_F is connected we obtain statement (iv), so we have proved Claim 5.2.

To continue the proof of Lemma 5.1, we say that an edge $e' \in M$ is *good* if $\mathbf{a}' \cdot \chi(e') \leq U + d\sqrt{\alpha}$. We say that a vertex $v \in V(J)$ is *good* if it lies in a good edge of M . Note that if $v_{i,j}$ is a good vertex and $v_{i,j} \in e' \in M$, then Claim 5.2 (i) gives $0 \leq (\mathbf{a}' - \mathbf{a}) \cdot \chi(e') \leq d\sqrt{\alpha}$, so $a'_{i,j} - a_{i,j} < d\sqrt{\alpha}$. Writing B for the set of bad edges of M , by Claim 5.2 (i) and (ii) we have $|B|d\sqrt{\alpha} < \sum_{e' \in B} (\mathbf{a}' \cdot \chi(e') - U) \leq \sum_{e' \in M} (\mathbf{a}' \cdot \chi(e') - U) \leq rd\alpha n$, so $|B| < r\sqrt{\alpha}n$. Thus all but at most $rk\sqrt{\alpha}n$ vertices $v \in V(J)$ are good. Next we need another claim.

Claim 5.3. *There is some $p \in [k-1]$ and an edge $e^* \in J_{k-p+1}$ in which all vertices are good, such that $\mathbf{a}' \cdot \chi(e^*) \geq (k+1)d\sqrt{\alpha} + \sum_{p \leq j \leq k} u_{f(j),j}$, where $f \in F$ is such that $e^* = \{v_p, \dots, v_k\}$ with $v_j \in V_{f(j)}$ for $p \leq j \leq k$.*

To prove this claim, we start by choosing p so that the gap $u_{1,p+1} - u_{1,p}$ is considerably larger than $u_{1,k} - u_{1,p+1}$. More precisely, we require

$$u_{1,p+1} - u_{1,p} > k(u_{1,k} - u_{1,p+1}) + (k+2)d\sqrt{\alpha}.$$

Suppose for a contradiction that this is not possible. Then for every $p \in [k-1]$ we have $u_{1,k} - u_{1,p} = (u_{1,k} - u_{1,p+1}) + (u_{1,p+1} - u_{1,p}) \leq (k+2)(u_{1,k} - u_{1,p+1} + d\sqrt{\alpha})$. Iterating this inequality starting from $u_{1,k} - u_{1,k-1} \leq (k+2)d\sqrt{\alpha}$ we obtain the bound $u_{1,k} - u_{1,p} \leq (k+3)^{k-p}d\sqrt{\alpha}$. However, for $p = 1$ we obtain $d = u_{1,k} - u_{1,1} \leq (k+3)^{k-1}d\sqrt{\alpha}$, which is a contradiction for $\alpha \ll 1/k$. Thus we can choose p as required. Now consider $S = \bigcup_{i \in [r], p < j \leq k} V_{i,j}$. Note that $|S \cap V_i| = (k-p)n/k + \alpha n$ for each $i \in [r]$. Thus we can apply the assumption that there is no space barrier, which gives at least βn^{k-p+1} edges in $J_{k-p+1}[S]$. At most $rk\sqrt{\alpha}n^{k-p+1}$ of these edges contain a vertex which is not good, so we may choose an edge

$e^* \in J_{k-p+1}$ whose vertices are all good and lie in S . Since J is $\mathcal{P}'F$ -partite, we can choose $f \in F$ so that $e^* = \{v_{f(p),\ell_p}, \dots, v_{f(k),\ell_k}\}$ for some ℓ_p, \dots, ℓ_k , and since $e^* \subseteq S$ we have $\mathbf{a}' \cdot \chi(e^*) \geq \sum_{p \leq j \leq k} u_{f(j),p+1}$. Then by Claim 5.2 (iv) we can estimate

$$\mathbf{a}' \cdot \chi(e^*) - \sum_{p \leq j \leq k} u_{f(j),j} \geq \sum_{p \leq j \leq k} (u_{1,p+1} - u_{1,j}) - D_F r k^2 d\alpha.$$

By choice of p we have $\sum_{p \leq j \leq k} (u_{1,p+1} - u_{1,j}) \geq (u_{1,p+1} - u_{1,p}) - (k-p)(u_{1,k} - u_{1,p+1}) > (k+2)d\sqrt{\alpha}$. The required bound on $\mathbf{a}' \cdot \chi(e^*)$ follows, so this proves Claim 5.3.

To finish the proof of Lemma 5.1, using the minimum degree sequence of J as above, we greedily extend e^* to an edge $e = \{v_{f(k),\ell_k}, \dots, v_{f(1),\ell_1}\} \in J_k$ with $\ell_j > jn/k - \alpha n$ for $j = p-1, p-2, \dots, 1$. This gives $\mathbf{a} \cdot \chi(e \setminus e^*) \geq \sum_{j \in [p-1]} u_{f(j),j}$. Also, since every vertex of e^* is good we have $(\mathbf{a}' - \mathbf{a}) \cdot \chi(e^*) \leq kd\sqrt{\alpha}$. It follows that

$$\begin{aligned} \mathbf{a} \cdot \chi(e) &= \mathbf{a}' \cdot \chi(e^*) - (\mathbf{a}' - \mathbf{a}) \cdot \chi(e^*) + \mathbf{a} \cdot \chi(e \setminus e^*) \\ &\geq (k+1)d\sqrt{\alpha} + \sum_{p \leq j \leq k} u_{f(j),j} - kd\sqrt{\alpha} + \sum_{j \in [p-1]} u_{f(j),j} = U_f + d\sqrt{\alpha}. \end{aligned}$$

On the other hand, for any $e' \in M$ we have $\mathbf{a} \cdot \chi(e) \leq \mathbf{a} \cdot \chi(e')$ by definition of A , and by Claim 5.2 (iii) we can choose e' so that $\mathbf{a} \cdot \chi(e') \leq U_f + D_F k d\alpha$. Thus we have a contradiction to our original assumption that $A \not\subseteq \{\chi(V_i) : i \in [r]\}$, which proves the Lemma. \square

Next we need to address a technical complication alluded to earlier, which is that we cannot satisfy the assumption $\delta \ll 1/n$ in Lemma 5.1 by working directly with J . Thus we need the following lemma, which will be proved by taking a random selection of edges in M , that allows us to reduce to a small matched subsystem (J', M') with similar properties to (J, M) . In the setting of the lemma, we say that M α -represents F if for any \mathbf{i}, \mathbf{i}' in $I(F)$, letting N, N' denote the number of edges $e' \in M$ with index \mathbf{i}, \mathbf{i}' respectively, we have $N' \geq (1 - \alpha)N$ (so each $\mathbf{i} \in I(F)$ is represented by approximately the same number of edges of M). Note that when (J, M) is $\mathcal{P}F$ -partite this implies that (J, M) is \mathcal{P} -proper (recall that this was a condition needed for Lemma 4.9.) Also, if M 0-represents F then the number of edges $e' \in M$ with $\mathbf{i}(e') = \mathbf{i}$ is the same for all $\mathbf{i} \in I(F)$; in this case we say that M is F -balanced.

Lemma 5.4. *Suppose that $1/n \ll 1/n' \ll \alpha' \ll \alpha \ll \beta' \ll \beta \ll 1/D_F, 1/r, 1/k$. Let \mathcal{P} partition a set V into r parts V_1, \dots, V_r each of size n . Suppose F is a (k, r) -uniform allocation with $|F| \leq D_F$ and (J, M) is a $\mathcal{P}F$ -partite matched k -system on V such that M α' -represents F . Suppose also that*

- (i) $\delta^F(J) \geq (n, (\frac{k-1}{k} - \alpha)n, (\frac{k-2}{k} - \alpha)n, \dots, (\frac{1}{k} - \alpha)n)$, and
- (ii) *for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ such that $|S_i| = \lfloor pn/k \rfloor$ for each $i \in [r]$ there are at least βn^{p+1} edges in $J_{p+1}[S]$, where $S := \bigcup_{i \in [r]} S_i$.*

Then for any $a \in [r]$ and $u, v \in V_a$, there exists a set $M' \subseteq M$ such that, writing $V' = \bigcup_{e \in M'} e$, $J' = J[V']$, and \mathcal{P}' for the partition of V' into $V'_i := V_i \cap V'$, $i \in [r]$, we have $u, v \in V'_a$, $|V'_1| = \dots = |V'_r| = n'$, and (J', M') is a $\mathcal{P}'F$ -partite matched k -system on V' such that

- (i) $\delta^F(J') \geq (n', (\frac{k-1}{k} - 2\alpha)n', (\frac{k-2}{k} - 2\alpha)n', \dots, (\frac{1}{k} - 2\alpha)n')$, and

- (ii) for any $p \in [k-1]$ and sets $S'_i \subseteq V'_i$ such that $|S'_i| = \lfloor pn'/k \rfloor$ for each $i \in [r]$ there are at least $\beta'(n')^{p+1}$ edges in $J'_{p+1}[S']$, where $S' := \bigcup_{i \in [r]} S'_i$.

The proof of this lemma requires some regularity theory, so we postpone it to the next section. Now we can combine Lemma 5.1 and Lemma 5.4, to obtain the following lemma, which shows that our assumptions guarantee irreducibility with the correct dependence of parameters. Note that the condition on M in the final statement is automatically satisfied if J is a complex.

Lemma 5.5. *Suppose that $1/n \ll 1/\ell \ll 1/B, 1/C \ll \alpha' \ll \alpha \ll \beta \ll 1/D_F, 1/r, 1/k$. Let V be a set of rn vertices, and let \mathcal{P}' partition V into r parts V_1, \dots, V_r each of size n . Suppose F is a (k, r) -uniform connected allocation with $|F| \leq D_F$, and (J, M) is a $\mathcal{P}'F$ -partite matched k -system on V such that M α' -represents F . Suppose also that*

- (i) $\delta^F(J) \geq (n, (\frac{k-1}{k} - \alpha)n, (\frac{k-2}{k} - \alpha)n, \dots, (\frac{1}{k} - \alpha)n)$, and
(ii) for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ such that $|S_i| = \lfloor pn/k \rfloor$ for each $i \in [r]$ there are at least βn^{p+1} edges in $J_{p+1}[S]$, where $S := \bigcup_{i \in [r]} S_i$.

Then (J_k, M) is (B, C) -irreducible with respect to \mathcal{P}' .

If, in addition, for any $v \in e' \in M$ we have $e' \setminus \{v\} \in J$, then there is a partition \mathcal{P} refining \mathcal{P}' such that $|U| \geq n/k - 2\alpha n$ and $D_\ell(J, M)[U]$ is complete for every $U \in \mathcal{P}$.

Proof. Choose n', δ with $1/B, 1/C \ll \delta \ll 1/n' \ll \alpha'$ and $kr \mid n'$, fix any $i \in [r]$, $u, v \in V_i$, and let M', V', J' be given by applying Lemma 5.4. Let \mathcal{P}'' be the partition of V' into $V' \cap V_1, \dots, V' \cap V_r$ and $X' = X(J', M') \subseteq \mathbb{R}^{n'}$. Applying Lemma 5.1, we have $B^{rn'}(\mathbf{0}, \delta) \cap \Pi_{\mathcal{P}''} \subseteq CH(X')$. Then by Lemma 4.1 we deduce that (J'_k, M') is (B, C) -irreducible with respect to \mathcal{P}'' . Since $i \in [r]$ and $u, v \in V_i$ were arbitrary, it follows that (J_k, M) is (B, C) -irreducible with respect to \mathcal{P}' . For the final statement we claim that $\delta^+(D_1(J_k, M)[V_q]) \geq \delta_{k-1}^F(J) \geq n/k - \alpha n$ for each $q \in [r]$. Indeed, consider any $v \in V_q$, and let e' be the edge of M containing v . Then we can write $e' = \{v_1, \dots, v_k\}$ where $v_k = v$ and $v_i \in V_{f(i)}$ for $i \in [k]$ for some $f \in F$. By assumption $e' \setminus \{v\} \in J$, so there are at least $\delta_{k-1}^f(J)$ vertices $u \in V_q$ such that $\{u\} \cup e' \setminus \{v\} \in J$. For each such u , $(\{\{u\} \cup e' \setminus \{v\}\}, \{e'\})$ is a simple (u, v) -transferral in (J_k, M) of size one, so this proves the claim. The conclusion then follows from Lemma 4.6. \square

Now we can formulate our main transferral lemma, which is that the minimum degree sequence assumption, with no divisibility or space barrier, implies the existence of all required simple transferrals. The proof is immediate from our previous lemmas.

Lemma 5.6. *Suppose that $1/n \ll 1/\ell \ll \alpha' \ll \alpha \ll \beta \ll 1/D_F, 1/r, 1/k$. Let V be a set of rn vertices, and let \mathcal{P}' be a partition of V into parts V_1, \dots, V_r each of size n . Suppose F is a (k, r) -uniform connected allocation with $|F| \leq D_F$, (J, M) is a $\mathcal{P}'F$ -partite matched k -system on V such that M α' -represents F , and for any $v \in e' \in M$ we have $e' \setminus \{v\} \in J$. Suppose also that*

- (i) $\delta^F(J) \geq (n, (\frac{k-1}{k} - \alpha)n, (\frac{k-2}{k} - \alpha)n, \dots, (\frac{1}{k} - \alpha)n)$,
(ii) for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ such that $|S_i| = \lfloor pn/k \rfloor$ for each $i \in [r]$ there are at least βn^{p+1} edges in $J_{p+1}[S]$, where $S := \bigcup_{i \in [r]} S_i$, and
(iii) $L_{\mathcal{P}}(J_k)$ is complete with respect to \mathcal{P}' for any partition \mathcal{P} of $V(J)$ which refines \mathcal{P}' and whose parts each have size at least $n/k - 2\alpha n$.

Then $D_\ell(J_k, M)[V_j]$ is complete for each $j \in [r]$. That is, for any $j \in [k]$ and $u, v \in V_j$ the matched k -graph (J_k, M) contains a simple (u, v) -transferral of size at most ℓ .

Proof. Let ℓ', B and C satisfy $1/\ell \ll 1/\ell' \ll 1/C, 1/B \ll \alpha$. By Lemma 5.5, (J_k, M) is (B, C) -irreducible with respect to \mathcal{P}' , and there is a partition \mathcal{P} refining \mathcal{P}' such that $|U| \geq n/k - 2\alpha n$ and $D_{\ell'}(J, M)[U]$ is complete for every $U \in \mathcal{P}$. Then $L_{\mathcal{P}}(J_k)$ is complete with respect to \mathcal{P}' by assumption, so Lemma 4.9 applies to give the required result. \square

6. HYPERGRAPH REGULARITY THEORY

This section contains the technical background that we need when applying the method of hypergraph regularity. Most of the machinery will be quoted from previous work, although we also give some definitions and short lemmas that are adapted to our applications. We start in the first subsection by introducing our notation, defining hypergraph regularity, and stating the ‘regular restriction lemma’. The second subsection states the Regular Approximation Lemma of Rödl and Schacht. In the third subsection we state the Hypergraph Blow-up Lemma, and prove a short accompanying result that is adapted to finding perfect matchings. In the fourth subsection we define reduced k -systems, and develop their theory, including the properties that they inherit degree sequences, and that their edges represent k -graphs to which the Hypergraph Blow-up Lemma applies. The final subsection contains the proof of Lemma 5.4, using the simpler theory of ‘weak regularity’ (as opposed to the ‘strong regularity’ in the rest of the section).

6.1. Hypergraph regularity. Let X be a set of vertices, and let \mathcal{Q} be a partition of X into r parts X_1, \dots, X_r . The *index* $i_{\mathcal{Q}}(S)$ of a set $S \subseteq X$ is the multiset in $[r]$ where the multiplicity of $j \in [r]$ is $i_{\mathcal{Q}}(S)_j = |S \cap X_j|$; we generally write $i(S) = i_{\mathcal{Q}}(S)$ when \mathcal{Q} is clear from the context. Recall that a set $S \subseteq X$ is \mathcal{Q} -partite if it has at most one vertex in each part of \mathcal{Q} ; for such sets $i(S)$ is a set. For any $A \subseteq [r]$ we write X_A for the set $\bigcup_{i \in A} X_i$, and $K_A(X)$ for the complete \mathcal{Q} -partite $|A|$ -graph on X_A , whose edges are all \mathcal{Q} -partite sets $S \subseteq X$ with $i(S) = A$. For a \mathcal{Q} -partite set $S \subseteq X$ and $A \subseteq i(S)$, we write $S_A = S \cap X_A$. If H is a \mathcal{Q} -partite k -graph or k -system on X , then for any $A \subseteq [r]$ we define $H_A := H \cap K_A(X)$. Equivalently, H_A consists of all edges of H with index A , which we regard as an $|A|$ -graph on vertex set X_A . If additionally H is a k -complex, then we write $H_{A \leq}$ for the k -complex on X_A with edge set $\bigcup_{B \subseteq A} H_B$, and $H_{A <}$ for the k -complex on X_A with edge set $\bigcup_{B \subset A} H_B$. Similarly, we write $H_{\mathbf{i}}$ for the set of edges in H with index vector \mathbf{i} . This is a $|\mathbf{i}|$ -graph, where $|\mathbf{i}| := \sum_{j \in [r]} i_j$.

To understand the definition of hypergraph regularity it is helpful to start with the case of graphs. If G is a bipartite graph on vertex classes U and V , we say that G is ε -regular if for any $U' \subseteq U$ and $V' \subseteq V$ with $|U'| > \varepsilon|U|$ and $|V'| > \varepsilon|V|$ we have $d(G[U' \cup V']) = d(G) \pm \varepsilon$. Similarly, for a k -complex G , an informal statement of regularity is that the restriction of G to any large subcomplex of the ‘lower levels’ of G has similar densities to G . Formally, let \mathcal{Q} partition a set X into parts X_1, \dots, X_r , and G be an \mathcal{Q} -partite k -complex on X . We denote by G_A^* the $|A|$ -graph on X_A whose edges are all those $S \in K_A(X)$ such that $S' \in G$ for every strict subset $S' \subset S$. So G_A^* consists of all sets which could be edges of G_A , in the sense that they are supported by edges at ‘lower levels’. The *relative density of G at A* is

$$d_A(G) := \frac{|G_A|}{|G_A^*|};$$

this is the proportion of ‘possible edges’ (given $G_{A<}$) that are in fact edges of G_A . (It should not be confused with the *absolute density of G at A* , which is $d(G_A) := |G_A|/|K_A(X)|$.)

For any $A \in \binom{[r]}{\leq k}$, we say that G_A is ε -regular if for any subcomplex $H \subseteq G_{A<}$ with $|H_A^*| \geq \varepsilon |G_A^*|$ we have

$$\frac{|G_A \cap H_A^*|}{|H_A^*|} = d_A(G) \pm \varepsilon.$$

We say G is ε -regular if G_A is ε -regular for every $A \in \binom{[r]}{\leq k}$.

The following lemma states that the restriction of any regular and dense k -partite k -complex to large subsets of its vertex classes is also regular and dense (it is a weakened version of [19, Theorem 6.18]).

Lemma 6.1. (*Regular restriction*) Suppose that $1/n \ll \varepsilon \ll c, 1/k$. Let \mathcal{Q} partition a set X into X_1, \dots, X_k , and G be an ε -regular \mathcal{Q} -partite k -complex on X with $d(G) \geq c$. Then for any subsets $X'_1 \subseteq X_1, \dots, X'_k \subseteq X_k$ each of size at least $\varepsilon^{1/2k}n$, the restriction $G' = G[X'_1 \cup \dots \cup X'_k]$ is $\sqrt{\varepsilon}$ -regular with $d(G') \geq d(G)/2$ and $d_{[k]}(G') \geq d_{[k]}(G)/2$.

6.2. The Regular Approximation Lemma. Roughly speaking, hypergraph regularity theory shows that an arbitrary k -graph can be split into pieces, each of which forms the ‘top level’ of a regular k -complex. To describe the splitting, we require the following definitions. Let \mathcal{Q} partition a set X into r parts X_1, \dots, X_r . A \mathcal{Q} -partition k -system on X consists of a partition \mathcal{C}_A of $K_A(X)$ for each $A \in \binom{[r]}{\leq k}$. We refer to the parts of each \mathcal{C}_A as *cells*, and to the cells of $\mathcal{C}_{\{i\}}$ for each $i \in [k]$ as the *clusters* of P . Observe that the clusters of P form a partition of X which refines \mathcal{Q} . For any \mathcal{Q} -partite set $S \subseteq X$ with $|S| \leq k$, we write C_S for the set of all edges of $K_{i(S)}(X)$ lying in the same cell of P as S . We write $C_{S \leq}$ for the \mathcal{Q} -partite k -system with vertex set X and edge set $\bigcup_{S' \subseteq S} C_{S'}$.

Let P be a \mathcal{Q} -partition k -system on X . We say that P is a \mathcal{Q} -partition k -complex if P satisfies the additional condition that whenever edges $S, S' \in K_A(X)$ lie in the same cell of \mathcal{C}_A , the edges S_B, S'_B of $K_B(X)$ lie in the same cell of \mathcal{C}_B for any $B \subseteq A$. Note that then $C_{S \leq}$ is a k -complex. We say that P is *vertex-equitable* if every cluster of P has the same size. We say that P is *a-bounded* if for every $A \in \binom{[r]}{\leq k}$ the partition \mathcal{C}_A partitions $K_A(X)$ into at most a cells. We say that P is ε -regular if $C_{S \leq}$ is ε -regular for every \mathcal{Q} -partite $S \subseteq X$ with $|S| \leq k$.

Now instead let P be a \mathcal{Q} -partition $(k-1)$ -complex on X . Then P naturally induces a \mathcal{Q} -partition k -complex P' on X . Indeed, for any $A \in \binom{[r]}{k}$ we say that $S, S' \in K_A(X)$ are *weakly equivalent* if for any strict subset $B \subset A$ we have that S_B and S'_B lie in the same cell of \mathcal{C}_B ; this forms an equivalence relation on $K_A(X)$. Then for each $A \in \binom{[r]}{\leq k-1}$ the partition \mathcal{C}'_A of P' is identical to the partition \mathcal{C}_A of P , and for each $A \in \binom{[r]}{k}$ the partition \mathcal{C}'_A of P' has the equivalence classes of the weak equivalence relation as its cells. We refer to P' as the \mathcal{Q} -partition k -complex generated from P by weak equivalence. Note that if P is a -bounded then P' is a^k -bounded, as for each $A \in \binom{[r]}{k}$ we have that $K_A(X)$ is divided into at most a^k cells by weak equivalence. Now let G be a \mathcal{Q} -partite k -graph on X . We denote by $G[P]$ the \mathcal{Q} -partition k -complex formed by using weak equivalence to refine the partition $\{G_A, K_A(X) \setminus G_A\}$ of $K_A(X)$ for each $A \in \binom{[r]}{k}$, i.e. two edges of G_A are in the same cell if they are weakly equivalent, and similarly for two k -sets in $K_A(X) \setminus G_A$. Together with P ,

this yields a partition k -complex which we denote by $G[P]$. If $G[P]$ is ε -regular then we say that G is *perfectly ε -regular with respect to P* .

We use the following form of hypergraph regularity due to Rödl and Schacht [39] (it is a slight reformulation of their result). It states that any given k -graph H can be approximated by another k -graph G that is regular with respect to some partition $(k-1)$ -complex P . It is convenient to consider the k -graph G , but we take care not to use any edges in $G \setminus H$, to ensure that every edge we use actually lies in the original k -graph H . There are various other forms of the regularity lemma for k -graphs which give information on H itself, but these do not have the hierarchy of densities necessary for the application of the blow-up lemma (see [19] for discussion of this point). In the setting of the theorem, we say that G and H are ν -close (with respect to \mathcal{Q}) if $|G_A \triangle H_A| < \nu |K_A(V)|$ for every $A \in \binom{[r]}{k}$.

Theorem 6.2. (*Regular Approximation Lemma*) *Suppose $1/n \ll \varepsilon \ll 1/a \ll \nu, 1/r, 1/k$ and $a!r|n$. Let V be a set of n vertices, \mathcal{Q} be a balanced partition of V into r parts, and H be a \mathcal{Q} -partite k -graph on V . Then there is an a -bounded ε -regular vertex-equitable \mathcal{Q} -partition $(k-1)$ -complex P on V , and an \mathcal{Q} -partite k -graph G on V , such that G is ν -close to H and perfectly ε -regular with respect to P .*

6.3. The hypergraph blowup lemma. While hypergraph regularity theory is a relatively recent development, still more recent is the hypergraph blow-up lemma due to Keevash [19], which makes it possible to apply hypergraph regularity theory to embeddings of spanning subcomplexes. Indeed, it is similar to the blow-up lemma for graphs, inasmuch as it states that by deleting a small number of vertices from a regular k -complex we may obtain a *super-regular* complex, in which we can find any spanning subcomplex of bounded maximum degree. However, unlike the graph case, the definition of super-regularity for complexes is extremely technical, so it is more convenient to work with a formulation using robustly universal complexes. In essence, a complex J' is robustly universal if even after the deletion of many vertices (with certain conditions), we may find any spanning subcomplex of bounded degree within the complex J that remains. The formal definition is as follows (we have simplified it by removing the option of ‘restricted positions’, which are not required in this paper).

Definition 6.3 (Robustly universal). *Suppose that V' is a set of vertices, \mathcal{Q} is a partition of V' into k parts $V'_1 \cup \dots \cup V'_k$, and J' is a \mathcal{Q} -partite k -complex on V' with $J'_{\{i\}} = V'_i$ for each $i \in [k]$. Then we say that J' is c -robustly D -universal if whenever*

- (i) $V_j \subseteq V'_j$ are sets with $|V_j| \geq c|V'_j|$ for each $j \in [k]$, such that writing $V = \bigcup_{j \in [k]} V_j$ and $J = J'[V]$ we have $|J_k(v)| \geq c|J'_k(v)|$ for any $j \in [k]$ and $v \in V_j$, and
- (ii) L is a k -partite k -complex of maximum vertex degree at most D whose vertex classes U_1, \dots, U_k satisfy $|U_j| = |V_j|$ for each $j \in [k]$,

then J contains a copy of L , in which for each $j \in [k]$ the vertices of U_j correspond to the vertices of V_j .

The following version of the hypergraph blow-up lemma states that we may obtain a robustly universal complex from a regular complex by deleting a small number of vertices (it is a special case of [19, Theorem 6.32]). After applying Theorem 6.2, we regard $Z = G \setminus H$ as the ‘forbidden’ edges of G ; so with this choice of Z in Theorem 6.4, the output $G' \setminus Z'$ is a subgraph of H .

Theorem 6.4. (*Blow-up Lemma*) Suppose $1/n \ll \varepsilon \ll d^* \ll d_a \ll \theta \ll d, c, 1/k, 1/D, 1/C$. Let V be a set of vertices, \mathcal{Q} be a partition of V into k parts V_1, \dots, V_k with $n \leq |V_j| \leq Cn$ for each $j \in [k]$, and G be an ε -regular \mathcal{Q} -partite k -complex on V such that $|G_{\{j\}}| = |V_j|$ for each $j \in [k]$, $d_{[k]}(G) \geq d$ and $d(G) \geq d_a$. Suppose $Z \subseteq G_k$ satisfies $|Z| \leq \theta|G_k|$. Then we can delete at most $2\theta^{1/3}|V_j|$ vertices from each V_j to obtain $V' = V'_1 \cup \dots \cup V'_k$, $G' = G[V']$ and $Z' = Z[V']$ such that

- (i) $d(G') > d^*$ and $|G'(v)_k| > d^*|G'_k|/|V'_i|$ for every $v \in V'_i$, and
- (ii) $G' \setminus Z'$ is c -robustly D -universal.

We will apply the blow-up lemma in conjunction with the following lemma for finding perfect matchings in subcomplexes of robustly universal complexes. The set X forms an ‘ideal’ for the perfect matching property, in that *any* extension W of X with parts of equal size contains a perfect matching. We will see later that a random choice of X has this property with high probability.

Lemma 6.5. Let \mathcal{Q} partition a set U into U_1, \dots, U_k , and G be a \mathcal{Q} -partite k -complex on U which is c -robustly 2^k -universal. Suppose we have $X_j \subseteq U_j$ for each $j \in [k]$ such that

- (i) $|X_j| \geq c|U_j|$ for each $j \in [k]$, and
- (ii) $|G_k[X \cup \{v\}](v)| \geq c|G_k(v)|$ for any $v \in U$, where $X := X_1 \cup \dots \cup X_k$.

Then for any sets W_j with $X_j \subseteq W_j \subseteq U_j$ for $j \in [k]$ and $|W_1| = \dots = |W_k|$, writing $W = W_1 \cup \dots \cup W_k$, the k -complex $G[W]$ contains a perfect matching,

Proof. By (i) we have $|W_j| \geq |X_j| \geq c|U_j|$ for each j , and by (ii) we have

$$|G_k[W](v)| \geq |G_k[X \cup \{v\}](v)| \geq c|G_k(v)|$$

for every $v \in W$. So by definition of a c -robustly 2^k -universal complex, G contains any k -partite k -complex on W with maximum vertex degree at most 2^k . In particular, this includes the complex obtained by the downward closure of a perfect matching in $K_{[k]}[W]$. \square

6.4. Reduced k -systems. Now we introduce and develop the theory of reduced k -systems, whose role in the k -system setting is analogous to that of reduced graphs in the graph setting. Informally speaking, an edge of the reduced system represents a set of clusters, which is dense in the original k -system J , and sparse in the forbidden k -graph Z . While this does not contain enough information for embeddings of general k -graphs, it is sufficient for matchings, which is our concern here. Note that this definition includes a partition $(k-1)$ -complex P , which will be obtained from the regularity lemma, but only the first level of this complex (the partition of X into clusters) is used in the definition.

Definition 6.6. Let \mathcal{Q} partition a set X of size n into parts of equal size, and let J be a \mathcal{Q} -partite k -system, Z a \mathcal{Q} -partite k -graph, and P a vertex-equitable \mathcal{Q} -partition $(k-1)$ -complex on X . Suppose $\nu \in \mathbb{R}$ and $\mathbf{c} = (c_1, \dots, c_k) \in \mathbb{R}^k$. We define the reduced k -system $R := R_{P\mathcal{Q}}^{JZ}(\nu, \mathbf{c})$ as follows.

Let V_1, \dots, V_m be the clusters of P , and let n_1 denote their common size. The vertex set of R is $[m]$, where vertex i corresponds to cluster V_i of P . Any partition \mathcal{P} of X which is refined by the partition of X into clusters of P naturally induces a partition of $[m]$, which we denote by \mathcal{P}_R : i and j lie in the same part of \mathcal{P}_R if and only if V_i and V_j are subsets of the same part of \mathcal{P} . Note that \mathcal{Q}_R partitions $[m]$ into parts of equal size.

The edges of R are defined as \emptyset , and those $S \in \binom{[m]}{j}$ for each $j \in [k]$ such that S is \mathcal{Q}_R -partite, $|J[\bigcup_{i \in S} V_i]| \geq c_j n_1^j$, and for any $S' \subseteq S$ of size j' , at most $\nu^{2-j'} n_1^{j'} n^{k-j'}$ edges of Z intersect each member of $\{V_i : i \in S'\}$.

Ideally, we would like the reduced k -system of a k -complex J to be itself a k -complex. However, it does not appear to be possible to define a reduced k -system that both accomplishes this and inherits a minimum degree condition similar to that of J . Instead, we have a weaker property set out by the following result, which shows that any subset of an edge of a reduced k -system of J is an edge of another reduced k -system of J , where the latter has weaker density parameters.

Lemma 6.7. *Let \mathcal{Q} partition a set X into parts of equal size, J be a \mathcal{Q} -partite k -system, Z a \mathcal{Q} -partite k -graph, and P a vertex-equitable \mathcal{Q} -partition $(k-1)$ -complex on X , with clusters of size n_1 . Suppose $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^k$ with $0 \leq c'_i \leq c_j$ for all $i, j \in [k]$ with $i \leq j$. Then for any $e \in R := R_{P\mathcal{Q}}^{JZ}(\nu, \mathbf{c})$ and $e' \subseteq e$ we have $e' \in R' := R_{P\mathcal{Q}}^{JZ}(\nu, \mathbf{c}')$.*

Proof. Since $e \in R$, e is \mathcal{Q}_R -partite, and so e' is \mathcal{Q}_R -partite, which is identical to being $\mathcal{Q}_{R'}$ -partite. Next, since $e \in R$, for any $e'' \subseteq e$ of size j at most $\nu^{2-j} n_1^j n^{k-j}$ edges of Z intersect each member of $\{V_i : i \in e''\}$. In particular, this property holds for any $e'' \subseteq e'$. Finally, since $e \in R$, we have $|J[\bigcup_{i \in e} V_i]| \geq c_{|e|} n_1^{|e|}$. Since J is a k -complex and $c'_{|e'|} \leq c_{|e|}$, it follows that $|J[\bigcup_{i \in e'} V_i]| \geq c'_{|e'|} n_1^{|e'|}$. \square

The next lemma shows that when the k -graph Z is sparse, we do indeed have the property mentioned above, namely that the minimum degree sequence of the original k -system J is ‘inherited’ by the reduced k -system. We work in the more general context of minimum F -degree sequences defined in Section 5.

Lemma 6.8. *Suppose that $1/n \ll 1/h, \nu \ll c_k \ll \dots \ll c_1 \ll \alpha, 1/k, 1/r$. Let X be a set of rn vertices, \mathcal{P} partition X into r parts X_1, \dots, X_r of size n , and \mathcal{Q} refine \mathcal{P} into h parts of equal size. Let J be a \mathcal{Q} -partite $(k-1)$ -system on X , Z a \mathcal{Q} -partite k -graph on X with $|Z| \leq \nu n^k$, and P a vertex-equitable \mathcal{Q} -partition $(k-1)$ -complex on X with clusters V_1, \dots, V_{rm} of size n_1 . Suppose also that J and Z are $\mathcal{P}F$ -partite for some allocation F , and that $\delta_0^F(J) = n$. Then, with respect to \mathcal{P}_R , the reduced k -system $R := R_{P\mathcal{Q}}^{JZ}(\nu, \mathbf{c})$ satisfies*

$$\delta^F(R) \geq \left((1 - k\nu^{1/2})m, \left(\frac{\delta_1^F(J)}{n} - \alpha \right) m, \dots, \left(\frac{\delta_{k-1}^F(J)}{n} - \alpha \right) m \right).$$

Proof. Let U_1, \dots, U_r be the parts of \mathcal{P}_R corresponding to X_1, \dots, X_r respectively. Fix any $f \in F$, $j \in [k-1]$ and $S = \{u_1, \dots, u_j\} \in R_j$ with $u_i \in U_{f(i)}$ for $i \in [j]$. Then S is \mathcal{Q}_R -partite and $\mathcal{P}_R F$ -partite, and $|J[V_S]| \geq c_j n_1^j$. For any edge $e \in J[V_S]$, we may write $e = \{v_1, \dots, v_j\}$ with $v_i \in X_{f(i)}$ for $i \in [j]$. There are at least $\delta_j^F(J)$ vertices $v_{j+1} \in X_{f(j+1)}$ such that $\{v_1, \dots, v_{j+1}\} \in J$, and of these at most jrn/h belong to the same part of \mathcal{Q} as one of v_1, \dots, v_j . Thus we obtain at least $|J[V_S]|(\delta_j^F(J) - jrn/h)$ edges in sets $J[V_T]$, $T \in \mathcal{T}$, where \mathcal{T} denotes the collection of \mathcal{Q}_R -partite sets $T = S \cup \{u\}$ for some $u \in U_{f(j+1)} \setminus S$. At most $mc_{j+1}n_1^{j+1} = c_{j+1}nn_1^j$ of these edges belong to sets $J[V_T]$ of size less than $c_{j+1}n_1^{j+1}$, so there are at least $\delta_j^F(J)m/n - \alpha m/2$ sets $T \in \mathcal{T}$ with $|J[V_T]| \geq c_{j+1}n_1^{j+1}$.

Such a set T is an edge of R , unless for some $T' \subseteq T$, more than $\nu^{2-j'-1} n_1^{j'+1} n^{k-j'-1}$ edges of Z intersect each part of $V_{T'}$, where $|T'| = j' + 1$ for some $0 \leq j' \leq j$; in this case we say that T is T' -bad. Note that such a set T' is not contained in S , otherwise S would be T' -bad, contradicting $S \in R$. So we can write $T' = S' \cup \{u\}$, where $S' \subseteq S$ and $T = S \cup \{u\}$. For fixed $S' \subseteq S$, there can be at most $\nu^{2-j'-1} m$ such vertices u , otherwise S would be S' -bad, contradicting $S \in R$. Summing over all $S' \subseteq S$ we find that there are at most $2^j \nu^{2-k} m$ vertices $u \in [U_{f(j+1)}]$ such that $T = S \cup \{u\}$ is T' -bad for some $T' \subseteq T$. It follows that $\delta_j^F(R) \geq \delta_j^F(J)m/n - \alpha m/2 - 2^j \nu^{2-k} m \geq \delta_j^F(J)m/n - \alpha m$.

It remains to show that $\delta_0^F(R) \geq (1 - k\nu^{1/2})m$, i.e. for each $i \in [r]$ which lies in the image of some $f \in F$, the set $\{u\}$ is an edge of R for all but at most $k\nu^{1/2}m$ vertices $u \in U_i$. For any $u \in U_i$, the set $\{u\}$ is \mathcal{Q}_R -partite, and since $\delta_0^f(J) = n$ we have $|J[V_u]| = n_1$. So $\{u\}$ is an edge of R unless more than $\nu^{1/2} n_1 n^{k-1}$ edges of Z intersect V_u ; since $|Z| \leq \nu n^k$ this is true for at most $k\nu^{1/2}m$ vertices u , as required. \square

The next proposition shows that the k -edges of the reduced k -system are useful, in that the corresponding clusters contain a sub- k -graph of J_k to which the blow-up lemma can be applied; here we note that the complex G' obtained in the proposition meets the conditions of Theorem 6.4 (with $\theta = \nu^{1/3}$ and d replaced by $d/2$).

Proposition 6.9. *Suppose that $1/n \ll \varepsilon \ll d_a \ll 1/a \ll \nu \ll d \ll 1/k$. Let X be a set of n vertices, \mathcal{Q} partition X into parts of equal size, G and Z be \mathcal{Q} -partite k -graphs on X , and P be a vertex-equitable a -bounded \mathcal{Q} -partition $(k-1)$ -complex on X , with clusters of size n_1 . Suppose that G is perfectly ε -regular with respect to P , and U_1, \dots, U_k are clusters of P such that $U := U_1 \cup \dots \cup U_k$ satisfies $|Z[U]| \leq \nu n_1^k$ and $|G[U]| \geq dn_1^k$. Let \mathcal{P} denote the partition of U into parts U_1, \dots, U_k . Then there exists a \mathcal{P} -partite k -complex G' on U such that $G'_k \subseteq G$, G' is ε -regular, $d_{[k]}(G') \geq d/2$, $d(G') \geq d_a$, and $Z' = Z \cap G'_k$ has $|Z'| \leq \nu^{1/3} |G'_k|$.*

Proof. To find G' we select a suitable cell of P^* , which we recall is the a^k -bounded \mathcal{Q} -partition k -complex formed from P by weak equivalence. Note that one of the partitions forming P^* is a partition of $K_{[k]}(U)$ into cells C_1, \dots, C_s , where $s \leq a^k$. So at most $dn_1^k/3$ edges of $K_{[k]}(U)$ lie within cells C_i such that $|C_i| \leq dn_1^k/(3a^k)$. Also, since $|Z[U]| \leq \nu n_1^k$, at most $\nu^{1/2} n_1^k$ edges of $K_{[k]}(U)$ lie within cells C_i such that $|Z \cap C_i| \geq \nu^{1/2} |C_i|$. Then, since $|G[U]| \geq dn_1^k$, at least $dn_1^k/2$ edges of G must lie within cells C_i with $|C_i| > dn_1^k/(3a^k)$ and $|Z \cap C_i| < \nu^{1/2} |C_i|$. By averaging, there must exist such a cell C_i that also satisfies $|G \cap C_i| > d|C_i|/2$. Fix such a choice of C_i , which we denote by C .

We now define G' to be the complex with top level $G'_k = G \cap C$ and lower levels $G'_{<k} = C_{<k} = \bigcup_{S' \subseteq S} C_{S'}$, where $S \in C$. Then $G'_k \subseteq G$, and G' is ε -regular, as G is perfectly ε -regular with respect to P . Furthermore, we have

$$d_{[k]}(G') = \frac{|G'_k|}{|(G')_{[k]}^*|} = \frac{|G \cap C|}{|C|} > d/2 \text{ and } d(G') = \frac{|G'_{[k]}|}{|K_{[k]}(U)|} = \frac{|G \cap C|}{|C|} \cdot \frac{|C|}{n_1^k} > \frac{d^2}{6a^k} > d_a.$$

Finally, since $|G'_k| = |G \cap C| > d|C|/2$ and $|Z'| \leq |Z \cap C| < \nu^{1/2} |C|$, we have $|Z'| < \frac{2\nu^{1/2}}{d} |G'_k| < \nu^{1/3} |G'_k|$. \square

Finally, we need the following lemma, which states (informally) that index vectors where the original k -system J is dense and the forbidden k -graph Z is sparse are inherited as index vectors where the reduced k -system R is dense. If J is a k -system on X , and \mathcal{P} partitions X , then we denote by $J_{\mathbf{i}}$ the set of edges in J with index \mathbf{i} with respect to \mathcal{P} .

Lemma 6.10. *Suppose that $1/n \ll \nu \ll c_k \ll \dots \ll c_1 \ll \mu, 1/k$. Let X be a set of n vertices, \mathcal{Q} partition X into parts of equal size, J be a \mathcal{Q} -partite k -system, Z a \mathcal{Q} -partite k -graph with $|Z| \leq \nu n^k$, and P a vertex-equitable \mathcal{Q} -partition $(k-1)$ -complex on X , with clusters V_1, \dots, V_m of size n_1 . Let $R := R_{PQ}^{JZ}(\nu, \mathbf{c})$ be the reduced k -system. Then for any partition \mathcal{P} of $V(J)$ which is refined by the partition of $V(J)$ into clusters of P we have the following property:*

(Inheritance of index vectors): *If $|J_{\mathbf{i}}| \geq \mu n^p$, where $p = |\mathbf{i}|$, then $|R_{\mathbf{i}}| \geq \mu m^p/2$, where index vectors are taken with respect to \mathcal{P} (for J) and \mathcal{P}_R (for R).*

Proof. Let \mathcal{B} be the set of \mathcal{Q}_R -partite sets S of size p with $S \notin R_p$. We estimate the number of edges contained in all p -graphs $J_p[V_S]$ with $S \in \mathcal{B}$. There are two reasons for which we may have $S \notin R_p$. The first is that $|J_p[V_S]| < c_p n_1^p$; this gives at most $c_p n^p$ edges in total. The second is that S is S' -bad for some $S' \subseteq S$, i.e. more than $\nu^{2-j'} n_1^{j'} n^{k-j'}$ edges of Z intersect each member of $\{V_i : i \in S'\}$, where $j' = |S'|$. For any given $j' \leq p$, there can be at most $\binom{p}{j'} \nu^{1/2} m^{j'}$ such sets S , otherwise we would have at least $\binom{p}{j'} \nu^{1/2} m^{j'}$ sets S' of size j' for which more than $\nu^{1/2} n_1^{j'} n^{k-j'}$ edges of Z intersect each member of $\{V_i : i \in S'\}$, contradicting $|Z| \leq \nu n^k$. Summing over $j' \leq p$, this gives at most $2^p \nu^{1/2} m^p n_1^p \leq c_p n^p$ edges in total. Thus there are at most $2c_p n^p$ edges contained in all p -graphs $J_p[V_S]$ with $S \in \mathcal{B}$. Now, if at least μn^p edges $e \in J_p$ have $\mathbf{i}_p(e) = \mathbf{i}$, then at least $\mu n^p/2$ of these edges lie in some $J_p[V_S]$ with $S \in R_p$. For each such S we have $\mathbf{i}_{\mathcal{P}_R}(S) = \mathbf{i}$ and $|J_p[V_S]| \leq n_1^p$. Thus there are at least $\mu m^p/2$ edges $S \in R_p$ with $\mathbf{i}_{\mathcal{P}_R}(S) = \mathbf{i}$. \square

6.5. Proof of Lemma 5.4. In this subsection it is more convenient to use the (much simpler) Weak Regularity Lemma, in the context of a simultaneous regularity partition for several hypergraphs. Suppose that \mathcal{P} partitions a set V into r parts V_1, \dots, V_r and G is a \mathcal{P} -partite k -graph on V . For $\varepsilon > 0$ and $A \in \binom{[r]}{k}$, we say that the k -partite sub- k -graph G_A is (ε, d) -vertex-regular if for any sets $V'_i \subseteq V_i$ with $|V'_i| \geq \varepsilon |V_i|$ for $i \in A$, writing $V' = \bigcup_{i \in A} V'_i$, we have $d(G_A[V']) = d \pm \varepsilon$. We say that G_A is ε -vertex-regular if it is (ε, d) -vertex-regular for some d . The following lemma has essentially the same proof as that of the Szemerédi Regularity Lemma [42], namely iteratively refining a partition until an ‘energy function’ does not increase by much; in this case the energy function is the sum of the mean square densities of the k -graphs with respect to the partition.

Theorem 6.11. *(Weak Regularity Lemma) Suppose that $1/n \ll 1/m \ll \varepsilon \ll 1/t, 1/r \leq 1/k$. Suppose that G^1, \dots, G^t are k -graphs on a set V of n vertices. Then there is a partition \mathcal{P} of V into $m' \leq m$ parts, such that there is some n_0 so that each part of \mathcal{P} has size n_0 or n_0+1 , and for each $i \in [t]$, all but at most εn^k edges of G^i belong to ε -vertex-regular k -partite sub- k -graphs.*

We also require the fact that regularity properties are inherited by random subsets with high probability. We use the formulation given by Czygrinow and Nagle [7, Theorem 1.2] (a similar non-partite statement was proved earlier by Mubayi and Rödl [31]).

Theorem 6.12. *Suppose that $1/n \ll 1/s, \varepsilon, c \ll \varepsilon', 1/k$, that \mathcal{P} partitions a set V into k parts V_1, \dots, V_r of size at least n , and G is an (ε, d) -vertex-regular k -partite k -graph on V . Suppose that $s_i \geq s$ for $i \in [k]$, and sets $S_i \subseteq V_i$ of size s_i are independently chosen uniformly at random for $i \in [k]$. Let $S = \bigcup_{i \in [k]} S_i$. Then $G[S]$ is (ε', d) -vertex-regular with probability at least $1 - e^{-c \min_i s_i}$.*

Here and later in the paper we will need the following inequality known as the Chernoff bound, as applied to binomial and hypergeometric random variables. We briefly give the standard definitions. The binomial random variable with parameters (n, p) is the sum of n independent copies of the $\{0, 1\}$ -valued variable A with $\mathbb{P}(A = 1) = p$. The hypergeometric random variable X with parameters (N, m, n) is defined as $X = |T \cap S|$, where $S \subseteq [N]$ is a fixed set of size m , and $T \subseteq [N]$ is a uniformly random set of size n . If $m = pN$ then both variables have mean pn .

Lemma 6.13. [17, Corollary 2.3 and Theorem 2.10] *Suppose X has binomial or hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

Now we give the proof of Lemma 5.4. We introduce new constants $\varepsilon, \varepsilon', c, d, m$ such that $1/n' \ll 1/m \ll \varepsilon, c \ll \varepsilon' \ll d \ll \alpha'$. Let e_u and e_v be the edges of M containing u and v respectively. Since M α' -represents F , we can partition M as $M_0 \cup M_1$, where M_1 is F -balanced, $\{e_u, e_v\} \subseteq M_1$ and $|M_0| \leq \alpha'|M|$. Letting V_0 be the set of vertices covered by M_0 , we have $|V_0| = k|M_0| \leq \alpha'rn$. Since $|M| = rn/k$ we can write $|M_1| = rn_1/k$ with $n_1 \geq (1 - \alpha')n$. By uniformity of F , M_1 covers n_1 vertices in each part of \mathcal{P} . Let $I := I(F)$ (recall that if $I(F)$ is a set then this is the set $\{i(f) : f \in F\}$). So $|I| = |F|/k!$. We partition M_1 into blocks, where each block contains one edge e' with $\mathbf{i}(e') = \mathbf{i}$ for each $\mathbf{i} \in I$. So each block consists of $|I|$ edges, and uses $b := k|I|/r$ vertices from each part. We arbitrarily label the blocks as B_i , $i \in [n_B]$, where $n_B := |M_1|/|I| = n_1/b$, and for each $j \in [r]$ we let B_{ij} be the vertices in V_j used by edges of B_i , which we arbitrarily label by $[b]$. For $x \in [n_B]$, $t \in [r]$, $c \in [b]$ we let v_{xtc} denote vertex c in B_{xt} . We construct M' as a union of blocks. We start by taking two distinct blocks B_{x_u} and B_{x_v} , where $e_u \in B_{x_u}$, and either $e_v \in B_{x_v}$ or $e_v \in B_{x_u}$ and B_{x_v} is arbitrary. Then we let X be a subset of $[n_B] \setminus \{x_u, x_v\}$ of size $n'/b - 2$ chosen uniformly at random, and let $X' = X \cup \{x_u, x_v\}$, so $|X'| = n'/b$. We take $M' = \bigcup_{x \in X'} B_x$. Writing $V' = \bigcup_{e \in M'} e$, $J' = J[V']$, and $V'_i = V_i \cap V'$ for each $i \in [r]$, we have $u, v \in V'_a$, $|V'_1| = \dots = |V'_r| = n'$, and (J', M') is a matched $\mathcal{P}F$ -partite k -system on V' .

Fix $f \in F$ and consider the minimum f -degree sequence $\delta^f(J')$. We clearly have $\delta_0^f(J') = n'$, as for any $v \in V'$ we have $\{v\} \in J$, and so $\{v\} \in J'$. Now fix any $e = \{v_1, \dots, v_j\} \in J$ for some $j \in [k-1]$ such that $v_i \in V_{f(i)}$ for $i \in [j]$. For each $i \in [j]$ let B_{d_i} be the block containing v_i , so $\mathcal{B}_e := B_{x_u} \cup B_{x_v} \cup \bigcup_{B_{d_i}} B_{d_i}$ is the union of $\ell \leq j+2$ distinct blocks. Let A be the set of vertices covered by edges in \mathcal{B}_e . Write $q = f(j+1)$, and for $c \in [b]$, let $N(e)_c$ be the set of vertices $v \in V_q \setminus (V_0 \cup A)$ such that $e \cup \{v\} \in J$ and $v = v_{xqc}$ for some $x \in [n_B]$. Then $\sum_{c \in [b]} |N(e)_c| \geq \left(\frac{k-j}{k} - \alpha\right)n - \alpha'rn - \ell br$ by assumption on $\delta^f(J)$. Writing $|N(e)_c| = \theta_c n_1$, and $C = \{c \in [b] : \theta_c > 2\varepsilon\}$ we have $\sum_{c \in C} \theta_c \geq 1 - j/k - \alpha - 2\alpha'r - 2b\varepsilon$. Note that, conditioning on the event $e \in J'$, the random variable $|N(e)_c \cap V'_q|$ is hypergeometric with mean $\frac{n'/b - \ell}{n_1/b - \ell} |N(e)_c| > (\theta_c - \varepsilon)n'$ for each $c \in C$, so the Chernoff bound (Lemma 6.13) gives $|N(e)_c \cap V'_q| > (\theta_c - 2\varepsilon)n'$ with probability at least $1 - 2e^{-\varepsilon^3 n'/3}$. On these events we have $m^f(e) := \sum_{c \in [b]} |N(e)_c \cap V'_q| > \sum_{c \in C} (\theta_c - 2\varepsilon)n' > (1 - j/k - 2\alpha)n'$. So, letting $Z(e, f)$

denote the event that $m^f(e) \leq (1 - j/k - 2\alpha)n'$, and no longer conditioning on $e \in J'$, we have

$$\mathbb{P}(e \in J' \text{ and } Z(e, f)) = \mathbb{P}(e \in J')\mathbb{P}(Z(e, f) \mid e \in J') \leq \left(\frac{n'/b - \ell}{n/b - \ell}\right)^{\ell-2} \cdot 2e^{-\varepsilon^3 n'/3}.$$

Taking a union bound over at most $|F|(k-1)(k+1)$ choices of $f \in F$, $j \in [k-1]$ and $\ell \in [j+1]$, and at most $(n_B)^{\ell-2}(b\ell)^j \leq n^{\ell-2}(2bk)^k$ edges $e \in J_j$ such that \mathcal{B}_e is the union of ℓ distinct blocks, we see that the minimum F -degree property of J' holds with high probability.

For property (ii) we use weak hypergraph regularity. Fix $p \in [k-1]$ and consider the following auxiliary $(p+1)$ -graphs J^{TC} on $[n_B]$. Given $1 \leq x_1 < \dots < x_{p+1} \leq n_B$, $T = (t_1, \dots, t_{p+1}) \in [r]^{p+1}$ such that there is some $f \in F$ for which $f(i) = t_i$ for $i \in [p+1]$, and $C = (c_1, \dots, c_{p+1}) \in [b]^{p+1}$, we say that $\{x_1, \dots, x_{p+1}\} \in J^{TC}$ if and only if $\{v_{x_i t_i c_i} : i \in [p+1]\} \in J_{p+1}$. Thus each edge of J_{p+1} that has at most one vertex in any block corresponds to a unique edge in at most $(p+1)!$ of the $(p+1)$ -graphs J^{TC} (these are given by holding C constant and permuting T). Observe that this accounts for most edges of J_{p+1} , as at most $br^{p+1}n^p$ edges have more than one vertex in some block. By Theorem 6.11, there is a partition \mathcal{P}' of $[n_B]$ into $m' \leq m$ parts, such that there is some n_0 so that each part of \mathcal{P}' has size n_0 or n_0+1 , and for each T and C as above, all but at most εn^{p+1} edges of J^{TC} belong to ε -vertex-regular \mathcal{P}' -partite sub- $(p+1)$ -graphs. Note that $m'n_0 \leq n_B \leq m'(n_0+1)$. Similarly to the previous argument, we will see that with high probability X' represents all parts of \mathcal{P}' approximately equally. Indeed, fix $U \in \mathcal{P}'$, and note that $|X' \cap U \setminus \{x_u, x_v\}|$ is hypergeometric with mean $\frac{n'/b-2}{n_1/b-2}(n_0 \pm 2) = \frac{n'}{n_1}(n_B/m' \pm 4) = n'/bm' \pm 1$, since $n_B = n_1/b$. By the Chernoff bound, we have $|X' \cap U| = (1 \pm \varepsilon)n'/bm'$ with probability at least $1 - 2e^{-\varepsilon^2 n'/6bm'}$. We restrict attention to choices of X' such that this estimate holds for all $U \in \mathcal{P}'$. Note that conditional on any specified values of $|X' \cap U|$ for $U \in \mathcal{P}'$, the choices of $X' \cap U \setminus \{x_u, x_v\}$, $U \in \mathcal{P}'$ are independent uniformly random $|X' \cap U \setminus \{x_u, x_v\}|$ -sets in $U \setminus \{x_u, x_v\}$ for $U \in \mathcal{P}'$. Now consider the ‘reduced’ $(p+1)$ -graphs R^{TC} , where $V(R^{TC}) = \mathcal{P}'$, and $E(R^{TC})$ consists of all $(p+1)$ -tuples (U_1, \dots, U_{p+1}) of parts of \mathcal{P}' , such that writing $U = \bigcup_{i \in [p+1]} U_i$, $J^{TC}[U]$ is (ε, d') -vertex-regular for some $d' \geq d$. Observe that at most $(d+\varepsilon)n^{p+1}$ edges of J^{TC} do not belong to $J^{TC}[U]$ for some such edge. Given an edge $(U_1, \dots, U_{p+1}) \in E(R^{TC})$, and a random choice of X' such that $|X' \cap U| = (1 \pm \varepsilon)n'/bm'$ for all $U \in \mathcal{P}'$, Theorem 6.12 implies that $J^{TC}[U \cap X']$ is (ε', d') -vertex-regular with probability at least $1 - e^{-cn'/2bm'}$. We can assume that this holds for all T, C and (U_1, \dots, U_{p+1}) , as there are at most $(rbm)^{p+1}$ choices, so we can take a union bound.

Now consider any sets $S'_t \subseteq V_t$ such that $|S'_t| = \lfloor pn'/k \rfloor$ for each $t \in [r]$. We need to show that there are at least $\beta'(n')^{p+1}$ edges in $J_{p+1}[S']$, where $S' := \bigcup_{t \in [r]} S'_t$. Let $S'_{tc} = \{x \in [n_B] : v_{xtc} \in S'\}$ for $t \in [r]$ and $c \in [b]$. Let $Y_{tc} = \{U \in \mathcal{P}' : |S'_{tc} \cap U| \geq \beta^2 |X' \cap U|\}$. For each $t \in [r]$ we have

$$pn'/k = |S'_t| = \sum_{c \in [b]} \sum_{U \in \mathcal{P}'} |S'_{tc} \cap U| \leq \sum_{c \in [b]} |Y_{tc}|(1 + \varepsilon)n'/bm' + b\beta^2 |X'|,$$

so $\sum_{c \in [b]} |Y_{tc}| > \frac{pn'/k - b\beta^2 n'}{(1+\varepsilon)n'/bm'} > (p/k - 2b\beta^2)bm'$. Write $S_t^0 = \bigcup_{c \in [b]} \bigcup_{U \in Y_{tc}} U$. Then $|S_t^0| > (p/k - 2b\beta^2)bm'n_0 > (p/k - 3b\beta^2)n$, since $m'(n_0+1) \geq n_B = n/b$. Let $S_t \subseteq V_t$ be an arbitrary set of size $\lfloor pn/k \rfloor$ such that $S_t^0 \subseteq S_t$ if $|S_t^0| \leq pn/k$ or $S_t \subseteq S_t^0$ if $|S_t^0| \geq pn/k$. Let

$S = S_1 \cup \dots \cup S_r$ and $S_{tc} = \{x \in [n_B] : v_{xtc} \in S\}$ for $t \in [r]$ and $c \in [b]$. By assumption (ii), there are at least βn^{p+1} edges in $J_{p+1}[S]$. Of these edges, we discard a small number of edges that are ‘bad’ for one of the following reasons: at most $3rk\beta^2 n^{p+1}$ edges which are incident with $\bigcup_{t \in [r]} (S_t \setminus S_t^0)$, at most $br^{p+1}n^p$ edges which have more than one vertex in some block, and at most $(br)^{p+1}(d + \varepsilon)n^{p+1}$ edges which correspond to an edge in some J^{TC} belonging to a $(p+1)$ -partite sub- $(p+1)$ -graph that is not (ε, d') -vertex-regular with $d' \geq d$. This still leaves at least $(\beta - \beta^{3/2})n^{p+1}$ edges, each of which corresponds to an edge $\{x_1, \dots, x_{p+1}\}$ in some J^{TC} such that $x_i \in S_{t_i c_i} \cap U_i$ for some $U_i \in Y_{t_i c_i}$ for $i \in [p+1]$ and $(U_i : i \in [p+1])$ is an edge of R^{TC} . Let Q^{TC} be the set of edges $(U_i : i \in [p+1])$ in R^{TC} such that $U_i \in Y_{t_i c_i}$ for $i \in [p+1]$. Then we have

$$\sum_{T,C} |Q^{TC}| \geq (\beta - \beta^{3/2})n^{p+1}/(n_0 + 1)^{p+1} \geq \frac{1}{2}\beta(bm')^{p+1}.$$

Recall that each edge of J with at most one vertex in any block corresponds to at most $(p+1)!$ edges in the $(p+1)$ -graphs J^{TC} . So $(p+1)!|J_{p+1}[S']|$ is at least the number of edges $\{x_1, \dots, x_{p+1}\}$ in some J^{TC} such that $x_i \in S'_{t_i c_i} \cap U_i$ for $i \in [p+1]$ where $(U_i : i \in [p+1]) \in Q^{TC}$. For each $(U_i : i \in [p+1]) \in Q^{TC}$ we have at least $(d - \varepsilon') \prod_{i \in [p+1]} |S'_{t_i c_i} \cap U_i| > \frac{1}{2}d(\beta^2 n'/bm')^{p+1}$ such edges of J^{TC} , using the definition of $Y_{t_i c_i}$ and ε' -vertex-regularity. Since $\sum_{T,C} |Q^{TC}| \geq \frac{1}{2}\beta(bm')^{p+1}$ we obtain at least $\beta'(n')^{p+1}$ edges in $J_{p+1}[S'] = J'_{p+1}[S']$. \square

7. MATCHINGS IN k -SYSTEMS

In this section we prove our theorems on matchings in k -systems with a minimum degree sequence condition. In fact, we prove theorems in the setting of minimum F -degree sequences, which simultaneously generalise both our non-partite and partite theorems. In the first subsection we prove the general form of the fractional perfect matching result. We combine this with hypergraph regularity in the second subsection to prove a common generalisation of Theorems 2.1 and 2.4 on almost perfect matchings. In the third subsection we apply transferrals to prove a common generalisation of Theorems 2.8 and 2.9 on perfect matchings; we will also see that essentially the same proof gives a common generalisation of Theorems 2.3 and 2.6.

7.1. Fractional perfect matchings. In this section we prove a lemma that generalises Lemma 3.6 to the minimum F -degree setting. Let J be a \mathcal{PF} -partite k -system on V , where \mathcal{P} is a balanced partition of V into r parts of size n , and F is a (k, r) -uniform allocation. Recall that a fractional perfect matching in J_k is an assignment of a weight $w_e \geq 0$ to each edge $e \in J_k$ such that for any $v \in V(J_k)$ we have $\sum_{e \ni v} w_e = 1$. The lemma will show that if J satisfies our minimum F -degree condition and has no space barrier then J_k admits a fractional perfect matching (in fact, with a slightly stronger minimum degree the latter condition is not required). We actually prove something stronger, namely that J_k contains a fractional perfect matching which is F -balanced, in that $\sum_{e \in J_k: \mathbf{i}(e)=\mathbf{i}} w_e$ is constant over all $\mathbf{i} \in I(F)$; this can be seen as a fractional equivalent of an F -balanced matching as previously defined. We also say that a multiset E in J_k is F -balanced if the number of edges in E of index \mathbf{i} (counted with multiplicity) is the same for any $\mathbf{i} \in I(F)$. First we need the following proposition. Let \mathcal{T} be the collection of all sets $T \subseteq J_k$ which contain one edge of index \mathbf{i} for each $\mathbf{i} \in I(F)$.

Proposition 7.1. *The following statements are equivalent.*

- (i) $\mathbf{1} \in PC(\{\chi(T) : T \in \mathcal{T}\})$.
- (ii) $k|I(F)|\mathbf{1}/rn \in CH(\{\chi(T) : T \in \mathcal{T}\})$.
- (iii) J_k admits an F -balanced fractional perfect matching in which at most $|I(F)|(rn + 1)$ edges have non-zero weight.
- (iv) J_k admits an F -balanced fractional perfect matching.

Proof. Let $X = \{\chi(T) : T \in \mathcal{T}\}$. Suppose first that (i) holds, so that $\mathbf{1} = \sum_{\mathbf{x} \in X} c_{\mathbf{x}} \mathbf{x}$ with $c_{\mathbf{x}} \geq 0$ for $\mathbf{x} \in X$. Then we have $k|I(F)| \sum_{\mathbf{x} \in X} c_{\mathbf{x}} = \sum_{\mathbf{x} \in X} c_{\mathbf{x}} \mathbf{x} \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{1} = rn$. Since any $\mathbf{x} \in X$ has non-negative integer coordinates we also have $c_{\mathbf{x}} \leq 1$ for each \mathbf{x} . Then by multiplying each $c_{\mathbf{x}}$ by $k|I(F)|/rn$ we obtain (ii). Now suppose that (ii) holds. Then by Theorem 3.1 we may write $k|I(F)|\mathbf{1}/rn = \sum_{\mathbf{x} \in X} c_{\mathbf{x}} \mathbf{x}$ with $c_{\mathbf{x}} \geq 0$ for $\mathbf{x} \in X$ so that at most $rn + 1$ of the $c_{\mathbf{x}}$'s are non-zero. For each $\mathbf{x} \in X$ assign weight $w_T = rnc_{\mathbf{x}}/k|I(F)|$ to some $T \in \mathcal{T}$ with $\chi(T) = \mathbf{x}$. Then assigning to each edge $e \in J_k$ the weight $w_e := \sum_{T \in \mathcal{T} : e \in T} w_T$ gives an F -balanced fractional perfect matching in which at most $|I(F)|(rn + 1)$ edges have non-zero weight, so we have (iii). Trivially (iii) implies (iv), so it remains to show that (iv) implies (i). Consider an F -balanced fractional perfect matching in J_k , where w_e denotes the weight of an edge e , so $\sum_{e \in J_k : \mathbf{i}(e) = \mathbf{i}} w_e$ is constant over $\mathbf{i} \in I(F)$. We assign weights to sets $T \in \mathcal{T}$ and modify the weights of edges $e \in J_k$ according to the following algorithm. Suppose at some step we have weights w'_e for $e \in J_k$, such that $\sum_{e \in J_k : \mathbf{i}(e) = \mathbf{i}} w'_e$ is constant over $\mathbf{i} \in I(F)$. Suppose that not all weights are zero, and choose $e_0 \in J_k$ with the smallest non-zero weight. For every $\mathbf{i} \in I(F)$ with $\mathbf{i} \neq \mathbf{i}(e_0)$, since $\sum_{e \in J_k : \mathbf{i}(e) = \mathbf{i}} w'_e = \sum_{e \in J_k : \mathbf{i}(e) = \mathbf{i}(e_0)} w'_e \neq 0$, we may choose $e_{\mathbf{i}}$ in J_k with $\mathbf{i}(e_{\mathbf{i}}) = \mathbf{i}$ and $w'_{e_{\mathbf{i}}} \neq 0$. Note that $w'_{e_{\mathbf{i}}} \geq w'_{e_0}$, by minimality of w'_{e_0} . Let $T \in \mathcal{T}$ consist of e_0 and the edges $e_{\mathbf{i}}$ for $\mathbf{i} \in I(F)$ with $\mathbf{i} \neq \mathbf{i}(e_0)$. We assign weight w'_{e_0} to T , and define new weights by $w''_{e_0} = 0$, $w''_{e_{\mathbf{i}}} = w'_{e_{\mathbf{i}}} - w'_{e_0}$, and $w''_e = w'_e$ otherwise. Then, with these new weights, $\sum_{e \in J_k : \mathbf{i}(e) = \mathbf{i}} w''_e$ remains constant over $\mathbf{i} \in I(F)$, and $w''_e \geq 0$ for every $e \in J_k$. Since the number of edges of zero weight has increased by at least one, after at most $|J_k|$ iterations every edge will have zero weight, at which point we stop. Then for any $\mathbf{x} \in X$, we let $c_{\mathbf{x}}$ be the sum of the weights assigned to any T with $\chi(T) = \mathbf{x}$. By construction we have $\mathbf{1} = \sum_{\mathbf{x} \in X} c_{\mathbf{x}} \mathbf{x}$, as required. \square

The following lemma generalises Lemma 3.6 to the minimum F -degree setting. Indeed, Lemma 3.6 is the $\alpha = 0$ ‘furthermore’ statement of Lemma 7.2, applied with $r = 1$ and F generated by the unique function $f : [k] \rightarrow [1]$. The main statement in Lemma 7.2 shows that the same conclusion holds under a slightly weaker F -degree sequence if there is no space barrier.

Lemma 7.2. *Suppose that $1/n \ll \alpha' \ll \beta, 1/D_F, 1/k, 1/r$ and that $\alpha \leq \alpha'$. Let V be a set partitioned into parts V_1, \dots, V_r each of size n , and F be a (k, r) -uniform connected allocation with $|F| \leq D_F$. Also let J be a \mathcal{PF} -partite k -system on V such that*

- (i) $\delta^F(J) \geq \left(n, \frac{(k-1)n}{k} - \alpha n, \frac{(k-2)n}{k} - \alpha n, \dots, \frac{n}{k} - \alpha n\right)$, and
- (ii) *for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ with $|S_i| = \lfloor pn/k \rfloor$ for $i \in [r]$ we have $|J_{p+1}[S]| \geq \beta n^{p+1}$, where $S := \bigcup_{i \in [r]} S_i$.*

Then J_k admits an F -balanced fractional perfect matching. Furthermore, if $\alpha = 0$ then this conclusion holds even without assuming condition (ii) or that F is connected.

Proof. First we give a construction that reduces to the case when $k \mid n$. For $i \in [r]$, let V'_i be a set of size kn consisting of copies $v_i(j)$, $j \in [k]$ of each $v_i \in V_i$. Let J' be the k -system whose edges are all possible copies of the edges of J . Then $\delta^F(J') = k\delta^F(J)$, so (i) holds for J' . To see that (ii) holds also, fix $p \in [k-1]$ and sets $S'_i \subseteq V'_i$ with $|S'_i| = pn$ for $i \in [r]$. Let $S_i \subseteq V_i$ consist of all vertices $v_i \in V_i$ with a copy $v_i(j)$ in S'_i . Then $|S_i| \geq |S'_i|/k \geq \lfloor pn/k \rfloor$. So $|J_{p+1}[S]| \geq \beta n^{p+1}$ by (ii), where $S := \bigcup_{i \in [r]} S_i$. Each edge of $J_{p+1}[S]$ has at least one copy in $J'_{p+1}[S']$, where $S' := \bigcup_{i \in [r]} S'_i$. We deduce that $|J'_{p+1}[S']| \geq \beta n^{p+1} \geq (\beta/k^{p+1})(kn)^{p+1}$, so (ii) holds for J' with β/k^{p+1} in place of β . Assuming the result when $k \mid n$, we find that J'_k admits an F -balanced fractional perfect matching. From this we obtain an F -balanced fractional perfect matching in J_k , where the weight of an edge is obtained by combining the weights of its copies in J'_k and dividing by k . Thus we can assume $k \mid n$.

Now suppose for a contradiction that J_k has no F -balanced fractional perfect matching. Let \mathcal{T} be as above; then Proposition 7.1 implies that $\mathbf{1} \notin PC(\{\chi(T) : T \in \mathcal{T}\})$. So by Farkas' Lemma (Lemma 3.4), there is some $\mathbf{a} \in \mathbb{R}^n$ such that $\mathbf{a} \cdot \mathbf{1} < 0$ and $\mathbf{a} \cdot \chi(T) \geq 0$ for every $T \in \mathcal{T}$. Note that any F -balanced multiset E in J_k can be expressed as $\sum_{i=1}^z T_i$ for some T_1, \dots, T_z in \mathcal{T} , and so satisfies $\mathbf{a} \cdot \chi(E) \geq 0$. For $i \in [r]$ let $v_{i,1}, \dots, v_{i,n}$ be the vertices of V_i , and let $a_{i,1}, \dots, a_{i,n}$ be the corresponding coordinates of \mathbf{a} , where the vertex labels are chosen so that $a_{i,1} \leq a_{i,2} \leq \dots \leq a_{i,n}$ for each $i \in [r]$.

For any multisets S and S' in V of equal size m , we say that S *dominates* S' , and write $S' \leq S$, if we may write $S = \{v_{i_1, j_1}, \dots, v_{i_m, j_m}\}$ and $S' = \{v_{i'_1, j'_1}, \dots, v_{i'_m, j'_m}\}$ so that for each $\ell \in [m]$ we have $i_\ell = i'_\ell$ and $j'_\ell \leq j_\ell$. Note that \leq is a transitive binary relation. We also observe that if $S' \leq S$ then $\mathbf{a} \cdot \chi(S') \leq \mathbf{a} \cdot \chi(S)$. As usual, for a multiset E in J_k we write $\chi(E) = \sum_{e \in E} \chi(e)$, thus identifying E with the multiset in V in which the multiplicity of $v \in V$ is the number of edges in E containing it, counting with repetition. We recall that $S + T$ denotes the multiset union of two multisets S and T , and for $i \in \mathbb{N}$, iS denotes the multiset union of i copies of S . We extend our 'arithmetic' of multisets to include subtraction, writing $S - T$ for the multiset A such that $A + T = S$, if it exists. It will also be convenient to manipulate formal expressions $S - T$ that do not correspond to multisets, via the rule $(S - T) + (S' - T') = (S + S') - (T + T')$, which can be understood as a shorthand for $\chi(S) - \chi(T) + \chi(S') - \chi(T') = \chi(S) + \chi(S') - \chi(T) + \chi(T')$.

We start by proving the $\alpha = 0$ statement. For a mental picture, it is helpful to think of the vertices arranged in a grid, with r columns corresponding to the parts V_i , $i \in [r]$, and k rows, where the subsquare in column i and row j contains the vertices $v_{i, (j-1)n/k + s}$, $s \in [n/k]$. We partition V into sets $(W_s : s \in [n/k])$, where $W_s = \{v_{i, (j-1)n/k + s} : i \in [r], j \in [k]\}$ consists of all vertices at *height* s , for each column i and row j . Note that for each $s \in [n/k]$ we have $W_s \geq W_1$. Thus we have $0 > \mathbf{a} \cdot \mathbf{1} = \sum_{s \in [n/k]} \mathbf{a} \cdot \chi(W_s) \geq (n/k) \mathbf{a} \cdot \chi(W_1)$. To obtain the required contradiction, we will show that a constant multiple of W_1 dominates an F -balanced multiset of edges of J_k . To see this, define $X^f := \{v_{f(j), (j-1)n/k + 1} : j \in [k]\}$ for $f \in F$. Since F is (k, r) -uniform we have $\sum_{f \in F} X^f = |F|r^{-1}W_1$. Now, by the minimum F -degree of J , we may greedily form an edge $e^f = \{v_{f(1), d_1}, \dots, v_{f(k), d_k}\} \in J$ with $d_1 = 1$ and $d_j \leq (j-1)n/k + 1$ for each $2 \leq j \leq k$. Then X^f dominates e^f , so $|F|r^{-1}W_1 = \sum_{f \in F} X^f$ dominates the F -balanced multiset $\{e^f : f \in F\}$. It follows that $\mathbf{a} \cdot \chi(W_1) \geq 0$, so we have the required contradiction to the assumption that J_k has no F -balanced fractional perfect matching.

Now consider the case $0 < \alpha < \alpha'$. We will obtain a contradiction by a similar strategy to that used when $\alpha = 0$, namely partitioning V into ‘dominating’ sets, where we say a multiset S is *dominating* if some constant multiple of S dominates an F -balanced multiset of edges in J_k . Note that if S is dominating we have $\mathbf{a} \cdot \chi(S) \geq 0$, so this will give the contradiction $\mathbf{a} \cdot \mathbf{1} \geq 0$. We may assume that $C := \alpha n$ is an integer. We also let N be an integer with $\alpha' \ll 1/N \ll 1/D_F, 1/k, 1/r$ and $|F| \mid N$. Now we define sets W_s , $s \in [n/k - CN]$ of size rk by

$$W_s := \{v_{i,s} : i \in [r]\} \cup \{v_{i,(j-1)n/k+C+s} : i \in [r], 2 \leq j \leq k\}.$$

Note that this agrees with our previous definition in the case when $C = \alpha = 0$; now we have increased by C the height of the vertices in rows 2 to k . Again we have $W_s \geq W_1$, and we will show that a constant multiple of W_1 dominates an F -balanced multiset of edges. To see this, define $X^f := \{v_{f(1),1}\} \cup \{v_{f(j),(j-1)n/k+C+1} : 2 \leq j \leq k\}$ for $f \in F$ (again, this agrees with our previous notation when $C = \alpha = 0$). Since F is (k, r) -uniform we have $\sum_{f \in F} X^f = |F|r^{-1}W_1$. By the minimum F -degree of J , we may greedily form an edge $e^f = \{v_{f(1),d_1}, \dots, v_{f(k),d_k}\} \in J$ with $d_1 = 1$ and $d_j \leq (j-1)n/k + C + 1$ for each $2 \leq j \leq k$. Then X^f dominates e^f , so $\sum_{f \in F} X^f = |F|r^{-1}W_1$ dominates $\{e^f : f \in F\}$.

Thus we have arranged that most vertices in V belong to dominating sets W_s , but in each column we still need to deal with the CN highest vertices in row 1, and the C lowest and $C(N-1)$ highest vertices in rows 2 to k . We will partition these into sets Z_s , $s \in [C]$ of size rkN , so that each Z_s contains N vertices in each of the rk subsquares of the grid of vertices, and in rows 2 to k these comprise 1 ‘low’ vertex and $N-1$ ‘high’ vertices. The formal definition is as follows. Define $P := [k] \times \{0, \dots, N-1\} \cup \{(1, N)\} \setminus \{(k, 0)\}$. Then for each $s \in [C]$, let

$$Z_s := \{v_{i,jn/k-tC+s} : (j, t) \in P, i \in [r]\}.$$

Note that the sets W_s , $s \in [n/k - CN]$ and Z_s , $s \in [C]$ partition V . Next we define a multiset that is dominated by each Z_s . Write $y_{i,j} = v_{i,jn/k-CN}$ for $i \in [r]$, $j \in [k]$,

$$Y = \{y_{i,j} : i \in [r], j \in [k]\}, \text{ and } D = NY + \{y_{i,1} : i \in [r]\} - \{y_{i,k} : i \in [r]\}.$$

Thus the CN highest vertices in the subsquare of column i and row j are each above the corresponding vertex $y_{i,j}$ of Y . We claim that each Z_s dominates D . To see this, note that for each $i \in [r]$ and $2 \leq j \leq k-1$, the N copies of $y_{i,j}$ in D are dominated by the $N-1$ high vertices $v_{i,jn/k-tC+s}$, $t \in [N-1]$ in row j and the low vertex $v_{i,jn/k+s}$ in row $j+1$. In row k we have removed one copy of each $y_{i,k}$ from D , so the remaining $N-1$ copies are dominated by the high vertices in row k . In row 1 we have $N+1$ copies of $y_{i,1}$ in D , which are dominated by the N vertices $v_{i,n/k-tC+s}$, $t \in [N]$ in row 1 and the low vertex $v_{i,n/k+s}$ in row 2. Thus $Z_s \geq D$.

The remainder of the proof is showing that D is a dominating multiset; we divide this into 3 claims. The first claim exploits the absence of a space barrier to find edges that are lower than those guaranteed by the minimum degree condition. Whereas the minimum degree condition gives edges in which the i th vertex is (close to being) in row i or below for $i \in [k]$, for each $p \in [k-1]$ we can find an edge where the first $p+1$ vertices are in row p or below, and the i th vertex is (close to being) in row i or below for $p+2 \leq i \leq k$. Intuitively, we can think of such an edge as having a ‘ p -demoted’ vertex, in that the $(p+1)$ st vertex is lower than guaranteed by the minimum degree condition (although the first p vertices may be higher). We also require that a demoted vertex is not too near the top of the row it has demoted too, in that it is below the corresponding vertex $y_{i,j}$. In the first claim we have

no control over which of the parts V_i contains a demoted vertex, so in the second claim we exploit the connectivity of F to construct a multiset with a demoted vertex in any desired part. Finally, in the third claim we take an appropriate non-negative linear combination of multisets with p -demoted vertices for all p to obtain a multiset dominated by D .

For the first claim we need edges dominated by one of the sets

$$B_p^f := \{y_{f(j),p} : j \in [p+1]\} \cup \{y_{f(j),j} : p+2 \leq j \leq k\}.$$

Claim 7.3. *For any $p \in [k-1]$ there is some $f_p \in F$ and $e_p \in J_k$ with $e_p \leq B_p^{f_p}$.*

To prove the claim, for each $i \in [r]$ let $S_i = \{v_{i,d} : d \leq pn/k - CN\}$, and arbitrarily choose S'_i of size pn/k containing S_i . Let $S = \bigcup_{i \in [r]} S_i$ and $S' = \bigcup_{i \in [r]} S'_i$. Then by condition (ii) we have $|J_{p+1}[S']| \geq \beta n^{p+1}$. At most $rCN(rn)^p < \beta n^{p+1}$ edges of J_{p+1} intersect $S' \setminus S$, so $|J_{p+1}[S]| > 0$. We may therefore choose an edge $e = \{v_{f(1),d_1}, \dots, v_{f(p+1),d_{p+1}}\} \in J_{p+1}$ where $d_j \leq pn/k - CN$ for $j \in [p+1]$, for some function $f : [p+1] \rightarrow [r]$. Since J is \mathcal{PF} -partite, f must be the restriction of some $f_p \in F$. Then by the minimum f_p -degree assumption on J we can greedily extend e to an edge $e_p = \{v_{f(1),d_1}, \dots, v_{f(k),d_k}\}$ with $d_j \leq (j-1)n/k + C + 1$ for $p+2 \leq j \leq k$. Thus $B_p^{f_p}$ dominates e_p , which proves Claim 7.3.

For the second claim we will find F -balanced multisets of edges dominated by the multisets

$$D_p^\ell = 2k|F|r^{-1}Y + (p+1)\{y_{\ell,p}\} - \{y_{\ell,j} : j \in [p+1]\}.$$

Claim 7.4. *For any $p \in [k-1]$ and $\ell \in [r]$ there is an F -balanced multiset E_p^ℓ in J_k with $E_p^\ell \leq D_p^\ell$.*

To prove the claim, we start by applying Claim 7.3, obtaining $e_p \in J_k$ and $f_p \in F$ with $e_p \leq B_p^{f_p}$. Since F is connected, there is a connected graph G_F on $[r]$ such that for every $ii' \in E(G_F)$ and $j, j' \in [k]$ with $j \neq j'$ there is $f \in F$ with $f(j) = i$ and $f(j') = i'$. Choose for each $j \in [p+1]$ a path $f_p(j) = i_1^j, \dots, i_{s_j+1}^j = \ell$ in G_F from $f_p(j)$ to ℓ . For each $z \in [s_j]$, let $f_z^j \in F$ be such that $f_z^j(j) = i_z^j$ and $f_z^j(p) = i_{z+1}^j$, and let \hat{f}_z^j be obtained from f_z^j by swapping the values of $f_z^j(j)$ and $f_z^j(p)$. Since F is invariant under permutation we have $\hat{f}_z^j \in F$. Now recall that for each $f \in F$ we have an edge $e^f \leq X^f$, where $X^f = \{v_{f(j), (j-1)n/k + C + 1} : j \in [k]\}$. We define $Y^f := \{y_{f(j),j} : j \in [k]\}$, and note that $Y^f \geq X^f \geq e^f$ and $\sum_{f \in F} Y^f = |F|r^{-1}Y$. Next we show that we can write D_p^ℓ as

$$(2) \quad D_p^\ell = 2k \sum_{f \in F} Y^f + (B_p^{f_p} - Y^{f_p}) + \sum_{j \in [p+1]} \sum_{z \in [s_j]} (Y^{f_z^j} - Y^{\hat{f}_z^j}).$$

To see this, note that $Y^{f_z^j} - Y^{\hat{f}_z^j} = \{y_{f_z^j(j),j}, y_{f_z^j(p),p}\} - \{y_{f_z^j(j),p}, y_{f_z^j(p),j}\} = (\{y_{i_z^j,j}\} - \{y_{i_z^j,p}\}) - (\{y_{i_{z+1}^j,j}\} - \{y_{i_{z+1}^j,p}\})$ for each $j \in [p+1]$, $z \in [s_j]$, so

$$\sum_{z \in [s_j]} (Y^{f_z^j} - Y^{\hat{f}_z^j}) = (\{y_{f_p(j),j}\} - \{y_{f_p(j),p}\}) - (\{y_{\ell,j}\} - \{y_{\ell,p}\})$$

for each $j \in [p+1]$. Then

$$\begin{aligned} \sum_{j \in [p+1]} \sum_{z \in [s_j]} (Y^{f_z^j} - Y^{\hat{f}_z^j}) &= \sum_{j \in [p+1]} (\{y_{f_p(j),j}\} - \{y_{f_p(j),p}\}) - \sum_{j \in [p+1]} (\{y_{\ell,j}\} - \{y_{\ell,p}\}) \\ &= Y^{f_p} - B_p^{f_p} + (p+1)\{y_{\ell,p}\} - \{y_{\ell,j} : j \in [p+1]\}, \end{aligned}$$

which proves (2). To define E_p^ℓ , we take $2k$ copies of e^f for each $f \in F$, replace one copy of e^{f_p} by a copy of e_p , and replace one copy of $e^{\hat{f}_z^j}$ by one copy of $e^{f_z^j}$ for each $j \in [p+1], z \in [s_j]$; note that there are enough copies for these replacements, as $f_z^j, z \in [s_j]$ are distinct. Thus E_p^ℓ contains $2k|F|$ edges and is F -balanced, since each edge was replaced by another of the same index. To see that $E_p^\ell \leq D_p^\ell$, we assign edges in E_p^ℓ to terms on the right hand side of (2), counted with multiplicity according to their coefficient, such that each edge of E_p^ℓ is dominated by its assigned term. Before the replacements, we have $2k$ copies of e^f for each $f \in F$, which we assign to the $2k$ copies of Y^f . To account for the replacement of a copy of e^{f_p} by e_p , we remove one of the assignments to Y_{f_p} and assign the replaced copy of e_p to $B_p^{f_p}$. To replace a copy of $e^{\hat{f}_z^j}$ by $e^{f_z^j}$, we remove one of the assignments to $Y^{\hat{f}_z^j}$ and assign the replaced copy of $e^{f_z^j}$ to $Y^{f_z^j}$. Thus we have an assignment showing that $E_p^\ell \leq D_p^\ell$, which proves Claim 7.4.

Finally, we will establish the required property of D by taking an appropriate non-negative linear combination of the multisets E_p^ℓ and D_p^ℓ . This is given by the following claim.

Claim 7.5. *There are non-negative integer coefficients $m_j, j \in [k]$ such that $M := \sum_{j=1}^k m_j \leq k^k$ and $D = (N - 2k|F|M)Y + \sum_{\ell \in [r]} \sum_{p \in [k-1]} m_p D_p^\ell$.*

To prove the claim, we define the coefficients recursively by $m_k = 0, m_{k-1} = 1, m_{k-2} = k - 1$ and

$$m_j = (j+2)m_{j+1} - (j+3)m_{j+2} \text{ for } 1 \leq j \leq k-3.$$

Since $m_j \leq km_{j+1}$ for any $j \in [k-1]$ we have $M \leq k^k$. Next we show that

$$(3) \quad (s+1)m_s - \sum_{j=s-1}^k m_j = \begin{cases} -1 & s = k, \\ 0 & k-1 \geq s \geq 2, \\ 1 & s = 1. \end{cases}$$

For $s = k$ we have $(k+1)m_k - m_{k-1} - m_k = -1$, and for $s = k-1$ we have $km_{k-1} - \sum_{j=k-2}^k m_j = k - (k-1) - 1 = 0$. Also, for $k-2 \geq s \geq 2$ we have

$$(s+1)m_s - \sum_{j=s-1}^k m_j = m_{s-1} + (s+2)m_{s+1} - \sum_{j=s-1}^k m_j = (s+2)m_{s+1} - \sum_{j=s}^k m_j,$$

so these cases follow from the case $s = k-1$. Finally, writing $m_0 = 0$, we have

$$\sum_{s=1}^k \left((s+1)m_s - \sum_{j=s-1}^k m_j \right) = m_k - m_0 = 0;$$

this implies the case $s = 1$, and so finishes the proof of (3). Now we show by induction that $m_j \geq (j+2)m_{j+1}/2$ for $j = k-2, k-3, \dots, 2$. For the base case $j = k-2$ we have

$m_{k-2} = k \geq k/2 = km_{k-1}/2$. Now suppose $2 \leq j \leq k-3$ and $m_{j+1} \geq (j+3)m_{j+2}/2$. Then $m_j - (j+2)m_{j+1}/2 \geq (j+2)m_{j+1}/2 - (j+3)m_{j+2} \geq (j-2)m_{j+1}/2 \geq 0$. Therefore $m_j \geq 0$ for $2 \leq j \leq k$. Then by (3), we also have $m_1 = (1 + \sum_{j=0}^k m_j) \geq 0$. Now by definition of D_p^ℓ , for any $\ell \in [r]$ we have

$$\begin{aligned} \sum_{p \in [k-1]} m_p D_p^\ell &= \sum_{p \in [k-1]} m_p (2k|F|r^{-1}Y + (p+1)\{y_{\ell,p}\} - \{y_{\ell,j} : j \in [p+1]\}) \\ &= 2Mk|F|r^{-1}Y + \sum_{s \in [k]} \left((s+1)m_s - \sum_{j=s-1}^k m_j \right) \{y_{\ell,s}\}. \end{aligned}$$

Summing over ℓ and applying (3), we obtain $\sum_{\ell \in [r]} \sum_{p \in [k-1]} m_p D_p^\ell = 2Mk|F|Y + \{y_{\ell,1} : \ell \in [r]\} - \{y_{\ell,k} : \ell \in [r]\}$. By definition of D , this proves Claim 7.5.

To finish the proof of the lemma, we show that D is dominating, by defining an F -balanced multiset E in J_k with $E \leq D$. We let E be the combination of $Nr/|F| - 2Mrk$ copies of $\{e^f : f \in F\}$ with m_p copies of E_p^ℓ for each $\ell \in [r]$ and $p \in [k]$ (note that $Nr/|F| - 2Mrk$ is positive since $N \gg D_F, k, r$). Then E is F -balanced, since each E_p^ℓ is F -balanced. Now recall from the proof of Claim 7.4 that we can write $\sum_{f \in F} Y^f = |F|r^{-1}Y$ with $Y^f \geq e^f$ for $f \in F$. Substituting this in the expression for D in Claim 7.5 we can see that D dominates E termwise: each Y^f in D dominates an e^f in E , and each D_p^ℓ in D dominates an E_p^ℓ in E . Now recall that V is partitioned into sets W_s , $s \in [n/k - CN]$ and Z_s , $s \in [C]$, where each W_s dominates the dominating set W_1 , and each Z_s dominates D , which we have now shown is dominating. We deduce that $\mathbf{a} \cdot \mathbf{1} \geq 0$, which gives the required contradiction to the assumption that J_k has no F -balanced fractional perfect matching. \square

7.2. Almost perfect matchings. In this subsection we prove a lemma that will be used in the following subsection to prove a common generalisation of Theorems 2.3 and 2.6, in which we find a matching covering all but a constant number of vertices, and a common generalisation of Theorems 2.8 and 2.9, where we find a perfect matching under the additional assumption that there is no divisibility barrier. The following lemma is a weaker version of the common generalisation of Theorems 2.3 and 2.6. It states that if a k -system J satisfies our minimum F -degree condition and does not have a space barrier, then J_k contains a matching which covers all but a small proportion of $V(J)$. One should note that the proportion ψ of uncovered vertices can be made much smaller than the deficiency α in the F -degree sequence. The proof will also show that a slightly stronger degree condition yields the same conclusion even in the presence of a space barrier, from which we shall deduce a common generalisation of Theorems 2.1 and 2.4 on almost perfect matchings.

Lemma 7.6. *Suppose that $1/n \ll \psi \ll \alpha \ll \beta, 1/D_F, 1/r, 1/k$. Let \mathcal{P} be a partition of a set V into parts V_1, \dots, V_r of size n and F be a connected (k, r) -uniform allocation with $|F| \leq D_F$. Suppose that J is a $\mathcal{P}F$ -partite k -system on V such that*

- (i) $\delta^F(J) \geq (n, (\frac{k-1}{k} - \alpha)n, (\frac{k-2}{k} - \alpha)n, \dots, (\frac{1}{k} - \alpha)n)$, and
- (ii) *for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ with $|S_i| = \lfloor pn/k \rfloor$ for $i \in [r]$ we have $|J_{p+1}[S]| \geq \beta n^{p+1}$, where $S := \bigcup_{i \in [r]} S_i$.*

Then J_k contains an F -balanced matching M which covers all but at most ψn vertices of J .

Proof. Introduce new constants with

$$1/n \ll \varepsilon \ll d_a \ll 1/a \ll \nu, 1/h \ll c_k \ll \dots \ll c_1 \ll \gamma \ll \psi \ll \alpha \ll \beta, 1/D_F, 1/r, 1/k.$$

We may additionally assume that $r \mid h$. Since $r \mid |V(J)|$ and $r \mid a!h$, we may delete up to $a!h$ vertices of J so that equally many vertices are deleted from each vertex class and the number of vertices remaining in each part is divisible by $a!h$. By adjusting the constants, we can assume for simplicity that in fact $a!h$ divides n , so no vertices were deleted. Fix any partition \mathcal{Q} of V into h parts of equal size which refines \mathcal{P} , and let J' be the k -system on V formed by all edges of J which are \mathcal{Q} -partite. Then J'_k is a \mathcal{Q} -partite k -graph on V , so by Theorem 6.2 there is an a -bounded vertex-equitable \mathcal{Q} -partition $(k-1)$ -complex P on V and a \mathcal{Q} -partite k -graph G on V that is ν/r^k -close to J'_k and perfectly ε -regular with respect to P . Let $Z := G \triangle J'_k$. Then since G is ν/r^k -close to J'_k we have $|Z| \leq \nu n^k$. Also let W_1, \dots, W_{rm_1} be the clusters of P (note that $rm_1 \leq ah$), and let n_1 be their common size, so $n_1 m_1 = n$.

Consider the reduced k -system $R := R_{P\mathcal{Q}}^{J'Z}(\nu, \mathbf{c})$. Recall that R has vertex set $[rm_1]$, where vertex i corresponds to cluster W_i of P , and that \mathcal{P}_R and \mathcal{Q}_R are the partitions of $[rm_1]$ corresponding to \mathcal{P} and \mathcal{Q} respectively. So R is $\mathcal{P}_R F$ -partite and \mathcal{Q} -partite. For each $i \in [r]$ let U_i be the part of \mathcal{P}_R corresponding to part V_i of \mathcal{P} ; since P was vertex-equitable each part U_i has size m_1 . We now show that since J had no space barrier, R also does not have a space barrier, even after the deletion of a small number of vertices from each U_i .

Claim 7.7. *Suppose that sets $U'_i \subseteq U_i$ for $i \in [r]$ satisfy $|U'_1| = \dots = |U'_r| = m' \geq (1-\alpha)m_1$. Let $U' := \bigcup_{i \in [r]} U'_i$, $R' := R[U']$ and $\mathcal{P}_{R'}$ be the restriction of \mathcal{P}_R to U' . Then for any $p \in [k-1]$ and sets $S'_i \subseteq U'_i$ with $|S'_i| = \lfloor pm'/k \rfloor$ for $i \in [r]$ we have $|R'_{p+1}[S']| \geq \beta(m')^{p+1}/10$, where $S' := \bigcup_{i \in [r]} S'_i$.*

To prove the claim, let $S = \bigcup_{i \in [r]} S_i$, where $S_i = \bigcup_{j \in S'_i} W_j \subseteq V_i \subseteq V$ for $i \in [r]$. Then $|S'_i| \geq n_1 \lfloor pm'/k \rfloor \geq (1-2\alpha)pn/k$ for $i \in [r]$. Let $S'' = \bigcup_{i \in [r]} S''_i$, where for $i \in [r]$ we take S''_i to be any set of size $\lfloor pn/k \rfloor$ such that $S_i \subseteq S''_i$ if $|S_i| \leq \lfloor pn/k \rfloor$ and $S''_i \subseteq S_i$ if $|S_i| \geq \lfloor pn/k \rfloor$. By assumption (ii) on J we have $|J_{p+1}[S'']| \geq \beta n^{p+1}$, and so $|J'_{p+1}[S'']| \geq \beta n^{p+1}/2$, since at most n^{p+1}/h edges of J_{p+1} are not edges of J'_{p+1} . Since at most $2\alpha rn$ vertices of S'' are not vertices of S , it follows that $|J'_{p+1}[S]| \geq \beta n^{p+1}/4$. Letting \mathcal{S} denote the partition of $V(J')$ into parts S and $V(J') \setminus S$, we can rephrase this as at least $\beta n^{p+1}/4$ edges $e \in J'_{p+1}$ have $\mathbf{i}_{\mathcal{S}}(e) = (p+1, 0)$. By Lemma 6.10 it follows that at least $\beta m_1^{p+1}/8$ edges $e \in R_{p+1}$ have $\mathbf{i}_{\mathcal{S}_R}(e) = (p+1, 0)$. Since \mathcal{S}_R is the partition of $[m_1]$ into S' and $[m_1] \setminus S'$, we conclude that $|R_{p+1}[S']| \geq \beta m_1^{p+1}/8$, and therefore $|R'_{p+1}[S']| \geq \beta(m')^{p+1}/10$, proving the claim.

Now, since any edge $e \in J'$ has $d_{J'}^F(e) \geq d_J^F(e) - \alpha n/2$, by Lemma 6.8 we have

$$\delta^F(R) \geq ((1 - k\nu^{1/2})m_1, (\delta_1(J)/n - \alpha/2)m_1, \dots, (\delta_{k-1}(J)/n - \alpha/2)m_1)$$

with respect to \mathcal{P}_R . So there are at most $k\nu^{1/2}m_1$ vertices i in each part of \mathcal{P}_R for which $\{i\}$ is not an edge of R . Thus we can delete $k\nu^{1/2}m_1$ vertices from each part to obtain R' with $m' = (1 - k\nu^{1/2})m_1 \geq (1 - \alpha)m_1$ vertices in each part and

$$(4) \quad \delta^F(R') \geq (m', (\delta_1(J)/n - \alpha)m', \dots, (\delta_{k-1}(J)/n - \alpha)m').$$

Claim 7.7 and (4) together show that R'_k satisfies the conditions of Lemma 7.2, and so admits an F -balanced fractional perfect matching. By Proposition 7.1 it follows that R'_k admits

a F -balanced fractional perfect matching in which there are at most $|I(F)|(rm' + 1)$ edges $e \in R'_k$ of non-zero weight. For $e \in R'_k$ let w_e be the weight of e in such a fractional matching. So $\sum_{e \ni v} w_e = 1$ for any $v \in V(R')$ and $\sum_{e \in R'_k: \mathbf{i}(e)=\mathbf{i}} w_e$ has common value $m'r/k|I(F)|$ for every $\mathbf{i} \in I(F)$. Next, partition each cluster W_i into parts $\{W_i^e : e \in R'_k\}$ such that

$$|W_i^e| = \begin{cases} w_e n_1 & \text{if } e \text{ is incident to vertex } i \text{ of } R', \\ 0 & \text{otherwise.} \end{cases}$$

The lemma will now follow easily from the following claim.

Claim 7.8. *For any $e \in R'_k$ there exists a matching M_e in $J_k[\bigcup_{i \in e} W_i^e]$ of size at least $(w_e - \gamma)n_1$.*

To prove the claim, first note that if $w_e \leq \gamma$ then there is nothing to prove, so we may assume that $w_e > \gamma$. Let M be a maximal matching in $J_k[\bigcup_{i \in e} W_i^e]$, and suppose for a contradiction that $|M| < (w_e - \gamma)n_1$. For each $i \in e$ let W'_i consist of the vertices in W_i^e not covered by M ; then $|W'_i| \geq \gamma n_1$ for each $i \in e$. Now observe that since $e \in R'_k$, and therefore $e \in R_k$, we know that e is \mathcal{Q}_R -partite, that $|Z[\bigcup_{i \in e} W_i^e]| \leq \nu^{2-k} n_1^k$, and that $|J'_k[\bigcup_{i \in e} W_i^e]| \geq c_k n_1^k$. Since $J'_k \setminus Z \subseteq G_k$ we therefore have $|G[\bigcup_{i \in e} W_i^e]| \geq c_k n_1^k/2$. Since G is perfectly ε -regular with respect to P , by Proposition 6.9 there is a k -partite k -complex G' whose vertex classes are W_i^e for $i \in e$ such that G' is ε -regular, $d_{[k]}(G') \geq c_k/4$, $d(G') \geq d_a$, $G'_k \subseteq G$, and $|Z \cap G'_k| \leq \nu^{2-k/3} |G'_k|$. Writing $W' := \bigcup_{i \in e} W'_i$, it follows by Lemma 6.1 that $G'[W']$ has $d(G'[W']) \geq d(G')/2$. So $G'_k[W']$ contains at least $\gamma^k |G'_k|/2$ edges. Since $\nu \ll \gamma$, some edge of $G'_k[W']$ is not an edge of Z , and is therefore an edge of $J_k[W']$, contradicting the maximality of M . This proves Claim 7.8.

To finish the proof of Lemma 7.6, we apply Claim 7.8 to find matchings M_e for each edge $e \in R'_k$ with $w_e > 0$. Then the union $M := \bigcup_{e \in R'_k: w_e > 0} M_e$ of all these matchings is a matching in J_k . Furthermore, by choice of the weights w_e , for any $\mathbf{i} \in I(F)$ the number of edges $e' \in M$ with $\mathbf{i}(e') = \mathbf{i}$ is at least

$$\begin{aligned} \sum_{e \in R'_k: w_e > 0, \mathbf{i}(e)=\mathbf{i}} (w_e - \gamma)n_1 &\geq \frac{rm'n_1}{k|I(F)|} - \gamma|I(F)|(rm' + 1)n_1 \\ &\geq \frac{(1 - k\nu^{1/2})rm_1n_1 - 2\gamma k D_F^2 rm_1n_1}{k|I(F)|} \geq \frac{|V(J)| - \psi n}{k|I(F)|}. \end{aligned}$$

So for each $\mathbf{i} \in I(F)$ we may choose $(|V(J)| - \psi n)/k|I(F)|$ edges $e' \in M$ with $\mathbf{i}(e') = \mathbf{i}$; these edges together form an F -balanced matching in J_k which covers all but at most ψn vertices of J . \square

Examining the above proof, we note that condition (ii) and the connectedness of F were only used in the proof of Claim 7.7, which in turn was only used to show that R'_k admits an F -balanced fractional perfect matching. If we instead assume the stronger F -degree condition $\delta^F(J) \geq (n, (\frac{k-1}{k} + \alpha)n, (\frac{k-2}{k} + \alpha)n, \dots, (\frac{1}{k} + \alpha)n)$ then we have $\delta^F(R') \geq (m', \frac{(k-1)m'}{k}, \dots, \frac{m'}{k})$ by (4). Then the ‘furthermore’ statement of Lemma 7.2 implies that R'_k admits an F -balanced fractional perfect matching. Thus we have also proved the following lemma.

Lemma 7.9. *Suppose that $1/n \ll \psi \ll \alpha \ll 1/D_F, 1/r, 1/k$. Let \mathcal{P} be a partition of a set V into parts V_1, \dots, V_r of size n and F be a (k, r) -uniform allocation with $|F| \leq D_F$. Suppose that J is a $\mathcal{P}F$ -partite k -system on V such that*

$$\delta^F(J) \geq \left(n, \left(\frac{k-1}{k} + \alpha \right) n, \left(\frac{k-2}{k} + \alpha \right) n, \dots, \left(\frac{1}{k} + \alpha \right) n \right).$$

Then J_k contains an F -balanced matching M which covers all but at most ψn vertices of J .

Now we deduce the common generalisation of Theorems 2.1 and 2.4. To do this, we add some ‘fake’ edges to increase the minimum degree of J , so that we may use Lemma 7.9. This gives an almost perfect matching in the new system. The fake edges are chosen so that only a small number of them can appear in any matching, so we then remove them to give an almost perfect matching in the original system.

Theorem 7.10. *Suppose that $1/n \ll \alpha \ll 1/D_F, 1/r, 1/k$. Let \mathcal{P} be a partition of a set V into sets V_1, \dots, V_r each of size n and F be a (k, r) -uniform allocation with $|F| \leq D_F$. Suppose that J is a $\mathcal{P}F$ -partite k -system on V with*

$$\delta^F(J) \geq \left(n, \left(\frac{k-1}{k} - \alpha \right) n, \left(\frac{k-2}{k} - \alpha \right) n, \dots, \left(\frac{1}{k} - \alpha \right) n \right).$$

Then J contains a matching covers all but at most $9k^2 r \alpha n$ vertices of J .

Proof. Choose a set X consisting of $8k\alpha n$ vertices in each part V_i for $i \in [r]$ uniformly at random. We form a k -system J' on $V(J)$ whose edge set consists of every edge of J , and *fake edges*, which are every $S \in \binom{V(J)}{\leq k}$ which is $\mathcal{P}F$ -partite and intersects X . We claim that with high probability we have

$$(5) \quad \delta^F(J') \geq \left(n, \left(\frac{k-1}{k} + \alpha \right) n, \left(\frac{k-2}{k} + \alpha \right) n, \dots, \left(\frac{1}{k} + \alpha \right) n \right).$$

To see this, first observe that $\delta_0^F(J') = \delta_0^F(J) = n$. Also, if e is a fake edge, then $d_{J'}^F(e) \geq n - k$. Now consider any $f \in F, j \in [k-1]$ and $e \in J_j$ such that we may write $e = \{v_1, \dots, v_j\}$ with $v_i \in V_{f(i)}$ for $i \in [j]$. Then there are at least $\left(\frac{k-j}{k} - \alpha \right) n$ vertices $v_{j+1} \in V_{f(j+1)}$ such that $\{v_1, \dots, v_{j+1}\} \in J$. We can assume that there is a set S of $n/2k$ vertices $v_{j+1} \in V_{f(j+1)}$ such that $\{v_1, \dots, v_{j+1}\} \notin J$ (otherwise we are done). Then $Y = |S \cap X|$ is hypergeometric with mean $4\alpha n$, so by Lemma 6.13 we have $Y \geq 2\alpha n$ with probability at least $1 - 2e^{-\alpha n/3}$. On this event, there are at least $(k-j)n/k + \alpha n$ vertices $v_{j+1} \in V_{f(j+1)}$ such that $\{v_1, \dots, v_{j+1}\} \in J'$. Taking a union bound over at most $|F|kn^{k-1}$ choices of f, j and e we see that (5) holds with high probability. Fix a choice of X such that (5) holds. Then by Lemma 7.9 J' contains a matching M' which covers all but at most αn vertices of J' . Since every fake edge intersects X , there can be at most $|X| = 8kr\alpha n$ fake edges in M' ; deleting these edges we obtain a matching M which covers all but at most $\alpha n + 8k^2 r \alpha n \leq 9k^2 r \alpha n$ vertices of J . \square

To deduce Theorem 2.1 we apply Theorem 7.10 with $r = 1$ and F consisting of $(k!)$ copies of the unique function $f : [k] \rightarrow [1]$, so $\delta^F(J) = \delta^f(J) = \delta(J)$. Similarly, to deduce Theorem 2.4 we apply Theorem 7.10 with F consisting of all injections $f : [k] \rightarrow [r]$, so $\delta^F(J) = \delta^*(J)$ is the partite minimum degree sequence.

7.3. Perfect matchings. In this subsection we prove a common generalisation of Theorems 2.8 and 2.9 on perfect matchings. Essentially the same proof will also give a common generalisation of Theorems 2.3 and 2.6. Before giving the proof, for the purpose of exposition we sketch an alternative argument for the case of graphs ($k = 2$). For simplicity we just consider the cases $r = 2$ and $r = 1$. Suppose first that $r = 2$, i.e. we have a bipartite graph G with parts V_1 and V_2 of size n , with $\delta(G) \geq (1/2 - \alpha)n$. Suppose that G does not have a perfect matching. Then by Hall's theorem, there is $S'_1 \subseteq V_1$ with $|N(S'_1)| < |S'_1|$. By the minimum degree condition, each of S'_1 and $N(S'_1)$ must have size $(1/2 \pm \alpha)n$. Let S_1 be a set of size $\lfloor n/2 \rfloor$ that either contains or is contained in S'_1 and let S_2 be a set of size $\lfloor n/2 \rfloor$ that either contains or is contained in $V_2 \setminus N(S'_1)$. Then $S_1 \cup S_2$ contains at most $2\alpha n^2$ edges, so we have a space barrier. Now suppose that $r = 1$, i.e. we have a graph G on n vertices with $\delta(G) \geq (1/2 - \alpha)n$, where n is even. Suppose that G does not have a perfect matching. Then by Tutte's Theorem, there is a set $U \subseteq V(G)$ so that $G \setminus U$ has more than $|U|$ odd components (i.e. connected components with an odd number of vertices). Suppose first that $|U| < (1/2 - 2\alpha)n$. Then $\delta(G \setminus U) \geq \alpha n$, so $G \setminus U$ has at most α^{-1} components. It follows that $|U| < \alpha^{-1}$. Then $\delta(G \setminus U) \geq (1/2 - 2\alpha)n$, so $G \setminus U$ has at most 2 components. It follows that $|U|$ is 0 or 1. Since n is even, $|U| = 0$, and G has two odd components, i.e. a divisibility barrier. Now suppose that $|U| > (1/2 - 2\alpha)n$. Then all but at most $2\alpha n$ of the odd components of $G \setminus U$ are isolated vertices, i.e. G contains an independent set I of size $(1/2 - 4\alpha)n$. Let S be a set of size $n/2$ containing I . Then S contains at most $2\alpha n^2$ edges, so we have a space barrier.

Theorem 7.11. *Let $1/n \ll \gamma \ll \alpha \ll \beta, \mu \ll 1/D_F, 1/r, 1/k$. Suppose F is a (k, r) -uniform connected allocation with $|F| \leq D_F$, and that $b = k|I(F)|/r$ divides n . Let \mathcal{P} be a partition of a set V into parts V_1, \dots, V_r of size n , and J be a $\mathcal{P}F$ -partite k -complex on V such that*

- (i) $\delta^F(J) \geq (n, (\frac{k-1}{k} - \alpha)n, (\frac{k-2}{k} - \alpha)n, \dots, (\frac{1}{k} - \alpha)n)$,
- (ii) *for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ such that $|S_i| = \lfloor pn/k \rfloor$ for each $i \in [r]$ there are at least βn^k edges of J_k with more than p vertices in $S := \bigcup_{i \in [r]} S_i$, and*
- (iii) $L_{\mathcal{P}}^\mu(J_k)$ is complete with respect to \mathcal{P} for any partition \mathcal{P}' of $V(J)$ which refines \mathcal{P} and whose parts each have size at least $n/k - \mu n$.

Then J_k contains a perfect matching which γ -represents F .

Proof. We follow the strategy outlined in Section 2.6. The first step is to use hypergraph regularity to decompose J into an exceptional set and some clusters with a matched reduced k -system. Introduce new constants with

$$\begin{aligned} 1/n \ll \varepsilon \ll d^* \ll d_a \ll 1/a \ll \nu, 1/h \ll \theta \ll d, c \ll c'_k \ll \dots \ll c'_1 \\ \ll c_k \ll \dots \ll c_1 \ll \psi \ll \gamma \ll 1/C \ll \alpha \ll \alpha' \ll \mu, \beta \ll 1/D_F, 1/r, 1/k. \end{aligned}$$

We also assume that $r \mid h$. By uniformity of F , any matching in J_k which contains one edge of index \mathbf{i} for each $\mathbf{i} \in I(F)$ covers b vertices in each part of \mathcal{P}' . Since $b \mid a!h$ and $b \mid n$, we can arbitrarily delete at most $a!h/b$ such matchings to make the number of vertices remaining in each part divisible by $a!h$. By adjusting the constants, we can assume for simplicity that in fact $a!h$ divides n , so no vertices were deleted. Fix any balanced partition \mathcal{Q} of V into h parts which refines \mathcal{P} , and let J' be the subcomplex obtained by deleting from J all those edges which are not \mathcal{Q} -partite. Then J'_k is a \mathcal{Q} -partite k -graph with order divisible by $a!h$, so by Theorem 6.2 there exists an a -bounded ε -regular vertex-equitable \mathcal{Q} -partition $(k-1)$ -complex P on $V(J)$ and a \mathcal{Q} -partite k -graph G on V that is ν/r^k -close to J'_k and perfectly

ε -regular with respect to P . Let $Z = G \triangle J'_k$, so $|Z| \leq \nu n^k$ and any edge of $G \setminus Z$ is also an edge of J_k . Note that since $r \mid h$ the number of clusters of P is divisible by r ; let W_1, \dots, W_{rm_1} be the clusters of P . Since P is a -bounded we must have $rm_1 \leq ah$. In addition, since P is vertex-equitable, each cluster W_i has the same size; let $n_1 := |W_1| = \dots = |W_{rm_1}| = n/m_1$ be this common size.

Let $R^1 := R_{P \setminus Z}^{J'_Z}(\nu, \mathbf{c})$ and $R^2 := R_{P \setminus Z}^{J'_Z}(\nu, \mathbf{c}')$ be reduced k -systems. So R^1 and R^2 have the common vertex set $[rm_1]$ partitioned into r parts U_1^1, \dots, U_r^1 by $\mathcal{P}_{R^1} = \mathcal{P}_{R^2}$, where for each $i \in [r]$ part U_i^1 of \mathcal{P}_{R^1} corresponds to part V_i of \mathcal{P} . Note that since P is vertex-equitable each U_i^1 has size m_1 . Also note that since R^2 has weaker density parameters than R^1 , any edge of R^1 is also an edge of R^2 . Now, since J is a k -complex, condition (ii) implies that for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ with $|S_i| = \lfloor pn/k \rfloor$ for $i \in [r]$ we have $|J_{p+1}[S]| \geq \beta n^{p+1}$, where $S := \bigcup_{i \in [r]} S_i$. So the conditions of Theorem 7.6 hold. Since R^1 is defined here exactly as R was in the proof of Theorem 7.6, we have the following claim, whose proof is identical to that of Claim 7.7.

Claim 7.12. *Suppose that sets $U_i \subseteq U_i^1$ for $i \in [r]$ satisfy $|U_1| = \dots = |U_r| = m \geq (1 - 2k\alpha)m_1$. Let $U := \bigcup_{i \in [r]} U_i$, $R := R^1[U]$ and let \mathcal{P}_R be the restriction of \mathcal{P}_{R^1} to U . Then for any $p \in [k-1]$ and sets $S_i \subseteq U_i$ with $|S_i| = \lfloor pm/k \rfloor$ for $i \in [r]$ we have $|R_{p+1}[S]| \geq \beta m^{p+1}/10$, where $S := \bigcup_{i \in [r]} S_i$.*

Since any edge $e \in J'$ has $d_{J'}^F(e) \geq d_J^F(e) - \alpha n/2$, by Lemma 6.8 we have

$$(6) \quad \delta^F(R^1), \delta^F(R^2) \geq \left((1 - k\nu^{1/2})m_1, \left(\frac{k-1}{k} - 2\alpha \right) m_1, \dots, \left(\frac{1}{k} - 2\alpha \right) m_1 \right).$$

So there are at most $k\nu^{1/2}m_1$ vertices i in each part of \mathcal{P}_{R^1} for which $\{i\}$ is not an edge of R^1 . We can therefore delete $k\nu^{1/2}m_1$ vertices from each part to obtain $R^0 \subseteq R^1$ with $m_0 = (1 - k\nu^{1/2})m_1$ vertices in each part and

$$\delta^F(R^0) \geq \left(m_0, \left(\frac{k-1}{k} - 3\alpha \right) m_0, \dots, \left(\frac{1}{k} - 3\alpha \right) m_0 \right).$$

By (6) and Claim 7.12, R^0 satisfies the conditions of Lemma 7.6, so contains an F -balanced matching M which covers all but at most ψm_0 vertices of R^0 . For each $i \in [r]$ let U_i be the vertices in the i th part of \mathcal{P}_{R^1} covered by M ; by uniformity of F each U_i has size $m \geq (1 - \psi/r)m_0 \geq (1 - 2\psi/r)m_1$. Let \mathcal{P}_R be the restriction of \mathcal{P}_{R^1} to $U := \bigcup_{i \in [r]} U_i$, so that the parts of \mathcal{P}_R are U_i for $i \in [r]$. Note that M is also a matching in R^2 . Let R be the restriction of R^2 to the vertices covered by M ; then (R, M) is a matched $\mathcal{P}_R F$ -partite k -system on U . For each edge $e \in M$, we arbitrarily relabel the clusters W_i for $i \in e$ as V_1^e, \dots, V_k^e , and let $V^e = \bigcup_{j=1}^k V_j^e$. Then the following claim provides the partition of $V(J)$ required for the first step of the proof.

Claim 7.13. *There is a partition of $V(J)$ into an exceptional set Γ and sets X, Y , where X, Y are partitioned into X^e, Y^e for $e \in M$, and X^e, Y^e are partitioned into sets X_j^e, Y_j^e , $j \in [k]$, such that*

- (i) Γ is partitioned into sets $\Gamma_1, \dots, \Gamma_t$ with $t \leq 3\psi n$, where each Γ_i has $b = k|I(F)|/r$ vertices in each V_i , $i \in [r]$,
- (ii) for any sets $X_j^e \subseteq \Lambda_j^e \subseteq X_j^e \cup Y_j^e$, $j \in [k]$ with $|\Lambda_1^e| = \dots = |\Lambda_k^e|$, writing $\Lambda^e = \Lambda_1^e \cup \dots \cup \Lambda_k^e$, there is a perfect matching in $J'[\Lambda^e]$,

- (iii) for any $j \in [r]$, $v \in V_j$, $\mathbf{i} \in I(F)$ with $i_j > 0$ and $S \subseteq V(J) \setminus X$ with $|S \cap V_i| \geq n/2 - n/6k$ for each i , there is an edge $e \in J[S \cup \{v\}]$ with $v \in e$ and $\mathbf{i}_P(e) = \mathbf{i}$, and
- (iv) $|X_j^e| = n_1/2$ and $|Y_j^e| = (1/2 - \psi/r)n_1$ for each $e \in M$ and $j \in [k]$.

To prove the claim, we start by applying Proposition 6.9 and Theorem 6.4 to each $e \in M$, deleting at most θn_1 vertices from each V_j^e to obtain $\hat{V}_1^e, \dots, \hat{V}_k^e$ and a k -partite k -complex G^e on $\hat{V}^e := \hat{V}_1^e \cup \dots \cup \hat{V}_k^e$ such that

- (i) $G_k^e \subseteq G \setminus Z \subseteq J_k$,
- (ii) G^e is c -robustly 2^k -universal, and
- (iii) $|G_k^e(v)| \geq d^* n^{k-1}$ for any $v \in \hat{V}^e$.

For each $j \in [k]$ choose $X_j^e \subseteq \hat{V}_j^e$ of size $n_1/2$ uniformly at random and independently of all other choices. Then Claim 7.13 (ii) holds with probability $1 - o(1)$: this follows from Lemma 6.5, for which condition (i) is immediate, and condition (ii) holds by the following lemma from [21] (it is part of Lemma 4.4), which is proved by a martingale argument.

Lemma 7.14. *Suppose that $1/n \ll d^*, b_2, 1/k, 1/b_1 < 1$. Let H be a k -partite k -graph with vertex classes V_1, \dots, V_k , where $n \leq |V_i| \leq b_1 n$ for each $i \in [k]$. Also suppose that H has density $d(H) \geq d^*$ and that $b_2 |V_i| \leq t_i \leq |V_i|$ for each i . If we choose a subset $X_i \subseteq V_i$ with $|X_i| = t_i$ uniformly at random and independently for each i , and let $X = X_1 \cup \dots \cup X_k$, then the probability that $H[X]$ has density $d(H[X]) > d^*/2$ is at least $1 - 1/n^2$.*

Next, for each $e \in M$, arbitrarily delete up to $\psi n_1/r$ vertices from each $\hat{V}_j^e \setminus X_j^e$ to obtain Y_j^e such that $|X_j^e \cup Y_j^e| = (1 - \psi/r)n_1$ for each $j \in [k]$. Note that Claim 7.13 (iv) is then satisfied. Let Γ consist of all vertices of J which do not lie in some X^e or Y^e . Then (Γ, X, Y) is a partition of $V(J)$. Note that Γ consists of all vertices in clusters W_j which were not covered by edges of M , and all vertices deleted in forming the sets X^e and Y^e . There are at most $2\psi m_1 \cdot n_1 = 2\psi n$ vertices of the first type, and at most $rm_1 \cdot \psi n_1/r = \psi n$ vertices of the second type, so $|\Gamma| \leq 3\psi n$. Also, b divides $|\Gamma \cap V_i|$ for $i \in [r]$, as b divides n , and M is F -balanced. (We also use the fact that each $X_j^e \cup Y_j^e$, $e \in M$, $j \in [k]$ has the same size.) We fix an arbitrary partition of Γ into $\Gamma_1, \dots, \Gamma_t$ that satisfies Claim 7.13 (i). It remains to satisfy Claim 7.13 (iii). Since $|X \cap V_i| \leq n/2$ for each i , applying the Chernoff bound, with probability $1 - o(1)$ we have $\delta_\ell^F(J[V(J) \setminus X]) \geq (k - \ell)n/3k$ for each $\ell \in [k - 1]$ (we omit the calculation, which is similar to others in the paper, e.g. that in Theorem 7.10.) Now consider any j, v, \mathbf{i} and S as in Claim 7.13 (iii). Starting with v , we greedily construct an edge $e \in J[S \cup \{v\}]$ with $\mathbf{i}_P(e) = \mathbf{i}$ and $v \in e$. This is possible since $|X \cap V_i|, |S \cap V_i| > n/2 - n/6k$ for each i , so $\delta_\ell^F(J[V(J) \setminus X]) \geq n/3k > |V_i \setminus (X \cup S)|$ for $\ell \in [k - 1]$ and $i \in [r]$. This completes the proof of Claim 7.13.

The second step of the proof is the following claim, which states that (R, M) contains small transferrals between any two vertices in any part of \mathcal{P}_R , even after deleting the vertices of a small number of edges of M .

Claim 7.15. *Let R' and M' be formed from R and M respectively by the deletion of the vertices covered by an F -balanced submatching $M_0 \subseteq M$ which satisfies $|M_0| \leq \alpha m$. Let $\mathcal{P}_{R'}$ be the partition of $V(R')$ into U'_1, \dots, U'_r obtained by restricting \mathcal{P}_R . Then $D_C(R', M')[U'_i]$ is complete for each $i \in [r]$.*

To prove the claim, we consider a matched k -system (R', M') formed in this manner and verify the conditions of Lemma 5.6. Note that each U'_i has size m' , where $m \geq m' \geq$

$(1 - k\alpha/r)m \geq (1 - 2k\alpha)m_1$, and that M' is F -balanced. Note also that if $e \in M'$, then $e \in M$, so $e \in R^1$. By Lemma 6.7 we therefore have $e \setminus \{v\} \in R^2$ for any $v \in e$; this was the reason for using two reduced systems. Since R' is the restriction of R^2 to the vertices of M' , we therefore have $e \setminus \{v\} \in R'$ for any $v \in e$. Thus by Lemma 5.6, it suffices to verify the following conditions:

- (a) $\delta^F(R') \geq (m', (\frac{k-1}{k} - \alpha')m', (\frac{k-2}{k} - \alpha')m', \dots, (\frac{1}{k} - \alpha')m')$,
- (b) for any $p \in [k-1]$ and sets $S_i \subseteq U'_i$ such that $|S_i| = pm'/k$ for each $i \in [r]$ there are at least $\beta(m')^{p+1}/10$ edges in $R'_{p+1}[S]$, where $S := \bigcup_{i \in [r]} S_i$, and
- (c) $L_{\mathcal{P}_{R'}}^*(R'_k)$ is complete with respect to $\mathcal{P}_{R'}$ for any partition $\mathcal{P}_{R'}^*$ of $V(R')$ which refines $\mathcal{P}_{R'}$ and whose parts each have size at least $m'/k - 2\alpha'm'$.

Condition (a) is immediate from (6), as R' was formed from R^2 by deleting at most $2k\alpha m_1$ vertices from each part including every vertex i for which $\{i\} \notin R^2$. For the same reason, since every edge of R^1 is also an edge of R^2 , by Claim 7.12 we have (b). So it remains to verify condition (c). Consider any partition $\mathcal{P}_{R'}^*$ of $V(R')$ refining $\mathcal{P}_{R'}$ into d parts U_1^*, \dots, U_d^* each of size at least $m'/k - 2\alpha'm'$. Form a partition $\mathcal{P}_{R^2}^{**}$ of $[rm_1]$ into d parts $U_1^{**}, \dots, U_d^{**}$, where for each $i \in [r]$ the at most $2k\alpha m_1$ vertices of $[rm_1] \setminus V(R')$ in the same part of \mathcal{P}_{R^2} as U_i^* are inserted arbitrarily among the parts of $\mathcal{P}_{R^2}^{**}$ in that part of \mathcal{P}_{R^2} . Then let $\mathcal{P}^\#$ be the partition of $V(J') = V(J)$ into d parts $U_j^\# := \bigcup_{i \in U_j^{**}} W_i$ for each j . Since any part of $\mathcal{P}_{R'}^*$ or $\mathcal{P}_{R^2}^{**}$ has size at least $m'/k - 2\alpha'm' \geq m_1/k - \mu m_1$, any part of $\mathcal{P}^\#$ has size at least $n/k - \mu n$. Then $L_{\mathcal{P}^\#}^\mu(J_k)$ is complete with respect to \mathcal{P} by assumption (iii) on J . Now consider any $\mathbf{i} \in L_{\mathcal{P}^\#}^\mu(J_k)$. By definition, at least μn^k edges $e \in J_k$ have $\mathbf{i}_{\mathcal{P}^\#}(e) = \mathbf{i}$. By Lemma 6.10 it follows that at least $\mu m_1^k/2$ edges $e \in R_k^2$ have $\mathbf{i}_{\mathcal{P}_{R^2}^{**}}(e) = \mathbf{i}$. Since R' was formed from R^2 by deleting at most $2k\alpha m_1$ vertices from each part, there is at least one edge of R'_k with $\mathbf{i}_{\mathcal{P}_{R'}^*}(e) = \mathbf{i}$, so $\mathbf{i} \in L_{\mathcal{P}_{R'}^*}^*(R'_k)$. This verifies condition (c), so we have proved Claim 7.15.

Continuing with the proof of Theorem 7.11, the third step is to find a matching covering the exceptional set Γ . We proceed through $\Gamma_1, \dots, \Gamma_t$ in turn, at each step covering a set Γ_i and using transferrals to rebalance the cluster sizes. Suppose we have chosen matchings E_1, \dots, E_{s-1} for some $s \in [t]$, where E_i covers the vertex set $V(E_i)$, with the following properties.

- (a) The sets $V(E_i)$ for $i \in [s-1]$ are pairwise-disjoint and have size at most brk^2C ,
- (b) $\Gamma_i \subseteq V(E_i) \subseteq \Gamma_i \cup Y$ for each $i \in [s-1]$, and
- (c) $|\bigcup_{i \in [s-1]} V(E_i) \cap Y_1^e| = \dots = |\bigcup_{i \in [s-1]} V(E_i) \cap Y_k^e| \leq n_1/6k$ for any $e \in M$.

Then the following claim will enable us to continue the process.

Claim 7.16. *There is a matching E_s in J satisfying properties (a), (b), and (c) with s in place of $s-1$.*

To prove the claim, we need to first remove any clusters that have been too heavily used by E_1, \dots, E_{s-1} . Note that $\sum_{i \in [s-1]} |V(E_i)| \leq brk^2Ct \leq 3brk^2C\psi n$. Thus there are at most $\alpha m_1/|I(F)|$ edges $e \in M$ such that $|\bigcup_{i \in [s-1]} V(E_i) \cap Y_j^e| \geq n_1/7k$ for some $j \in [k]$. Choose arbitrarily an F -balanced matching M_0 in R of size at most αm_1 which includes every such $e \in M$, and let (R', M') be the matched k -system obtained by deleting the vertices covered by M_0 from both R and M . Also let $\mathcal{P}_{R'}$ be the partition of $V(R')$ into U'_1, \dots, U'_r obtained by restricting \mathcal{P}_R . Then by Claim 7.15, $D_C(R', M')[U'_i]$ is complete for each $i \in [r]$. Let $Y' \subseteq Y$ consist of all vertices of Y except for those which lie in clusters deleted in forming R'

and those which lie in some $V(E_i)$. Then $|Y' \cap V_\ell| \geq |Y \cap V_\ell| - k\alpha m_1 n_1 - tbrk^2C \geq n/2 - n/8k$ for any $\ell \in [k]$.

Now choose for every $x \in \Gamma_s$ an edge $e_x \in J$ with $x \in e_x$ so that $\{e_x : x \in \Gamma_s\}$ is an F -balanced matching in J . Since $|\Gamma_s \cap V_i| = b = k|I(F)|/r$ for each i , by uniformity of F this will be accomplished by including k edges e_x of index \mathbf{i} for each $\mathbf{i} \in I(F)$. By repeatedly applying Claim 7.13 (iii) we may choose these edges to lie within $Y' \cup \Gamma_s$ and to be pairwise-disjoint. The matching $\{e_x : x \in \Gamma_s\}$ then covers every vertex of Γ_s , but we may have unbalanced some clusters. To address this, consider the extra vertices $Q := \bigcup_{x \in \Gamma_s} e_x \setminus \{x\}$ that were removed. Since each $\mathbf{i} \in I(F)$ was represented k times, $Q_j := Q \cap V_j$ has size $(k-1)b$ for each $j \in [r]$. Divide each Q_j arbitrarily into b parts of size $k-1$, and for each $\mathbf{i} \in I(F)$ form $Q_{\mathbf{i}}$ of size $k(k-1)$ by taking i_j parts in Q_j , so that $(Q_{\mathbf{i}} : \mathbf{i} \in I(F))$ partitions Q and for each $\mathbf{i} \in I(F)$ the set $Q_{\mathbf{i}}$ has index $\mathbf{i}(Q_{\mathbf{i}}) = (k-1)\mathbf{i}$.

Next, for each $\mathbf{i} \in I(F)$ we arbitrarily pick a ‘target’ edge $e(\mathbf{i}) \in M'$ of index \mathbf{i} to which we will transfer the imbalances caused by $Q_{\mathbf{i}}$. Then for any $\mathbf{i} \in I(F)$ and $x \in Q_{\mathbf{i}}$ we may choose $i(x) \in [m_1]$ and $i'(x) \in e(\mathbf{i})$ such that x is in the cluster $W_{i(x)}$, and $i(x)$ and $i'(x)$ are in the same part of $\mathcal{P}_{R'}$. Furthermore, we may choose the $i'(x)$ ’s so that the multiset $\{i'(x) : x \in Q_{\mathbf{i}}\}$ contains $k-1$ copies of each vertex of $e(\mathbf{i})$. Since $D_C(R', M')[U'_i]$ is complete for each $i \in [r]$, there is a simple $(i'(x), i(x))$ -transferral of size at most C in (R', M') . Choose such a transferral for every $x \in Q$, and let (T, T') be the combination of these transferrals. To implement the transferral we need to select a matching $E_s^* = \{e^* : e \in T\}$ in $J[Y]$, whose edges correspond to the edges of T (counted with multiplicity), in that e^* contains one vertex in each cluster W_i with $i \in e$ for each $e \in T$. We can construct E_s^* greedily, using Lemma 6.1, since by Claim 7.13 (iv) Y has $n_1/2 - \psi n_1/r$ vertices in each cluster, and we have used at most $n_1/5k$ of these in E_1, \dots, E_{s-1} and edges so far chosen for E_s^* .

Now we let $E_s = \{e_x\}_{x \in \Gamma_s} \cup E_s^*$. Then $|E_s| = br + |T| \leq br + |Q|C \leq brkC$, so E_s covers at most brk^2C vertices. By construction E_s is disjoint from E_1, \dots, E_{s-1} and satisfies $\Gamma_s \subseteq V(E_s) \subseteq \Gamma_s \cup Y$. Furthermore, E_s avoids all clusters in which $\bigcup_{i \in [s-1]} V(E_i)$ covers at least $n_1/7k$ vertices. Thus it remains to show that $|V(E_s) \cap Y_1^e| = \dots = |V(E_s) \cap Y_k^e|$ for any $e \in M$. To see this, note that the transferral (T, T') was chosen so that $\chi(T) - \chi(T') = \sum_{x \in Q} \chi(\{i'(x)\}) - \chi(\{i(x)\}) \in \mathbb{R}^{m_1}$. For each edge $e \in E_s$ we write $\chi^R(e) \in \mathbb{R}^{m_1}$ for the vector with $\chi^R(e)_i = |e \cap W_i|$. (Note that vertices in the exceptional set do not contribute here.) We also have $\sum_{e \in E_s^*} \chi^R(e) = \chi(T)$ and $\sum_{x \in \Gamma_s} \chi^R(e_x) = \sum_{x \in Q} \chi(\{i(x)\})$. It follows that $\sum_{e \in E_s} \chi^R(e) = \chi(T') + \sum_{x \in Q} \chi(\{i'(x)\}) = \chi(T') + \sum_{\mathbf{i} \in I(F)} (k-1)\chi(e(\mathbf{i}))$. Since T' and $T' + (k-1)\sum_{\mathbf{i} \in I(F)} e(\mathbf{i})$ are multisets in M , this establishes the required property of E_s to prove Claim 7.16.

Thus we can greedily complete the third step of covering the exceptional set Γ , in such a way that for each $e \in M$ we use an equal number of vertices in each Y_j^e , $j \in [k]$. To finish the proof of the theorem, for each $e \in M$ and $j \in [k]$ let Λ_j^e consist of those vertices of $X_j^e \cup Y_j^e$ which are not covered by any of the sets $V(E_i)$ for $i \in [t]$. Then $X_j^e \subseteq \Lambda_j^e \subseteq X_j^e \cup Y_j^e$ and $|\Lambda_1^e| = \dots = |\Lambda_k^e|$. By Claim 7.13 (ii), writing $\Lambda^e = \Lambda_1^e \cup \dots \cup \Lambda_k^e$, there is a perfect matching in $J'_k[\Lambda^e]$. Combining these matchings for $e \in M$ and the matchings E_1, \dots, E_t we obtain a perfect matching M^* in J_k . Furthermore, all but at most $tbrkC \leq \gamma rn/kD_F^2$ edges of this matching lie in $J'[\Lambda^e]$ for some $e \in M$. Since M is F -balanced, it follows that M^* γ -represents F . \square

To deduce Theorem 2.8, we assume that properties 2 (Space barrier) and 3 (Divisibility barrier) do not hold, and show that property 1 (Matching) must hold. We will apply Theorem 7.11 with $r = 1$ and F consisting of $(k!)$ copies of the unique function $f : [k] \rightarrow [1]$, so that $\delta^F(J) = \delta^f(J) = \delta(J)$ and $b = k|I(F)|/r = k$ divides n . Now the conditions of Theorem 7.11 are satisfied: (i) by the minimum degree sequence, (ii) because there is no space barrier, and (iii) because there is no divisibility barrier. Then J_k contains a perfect matching, so this proves Theorem 2.8.

Similarly, we can deduce Theorem 2.9 for the case where $b = k \binom{r}{k}/r = \binom{r-1}{k-1}$ divides n by taking F to consist of all injective functions $f : [k] \rightarrow [r]$; then $\delta^F(J) = \delta^*(J)$ is the partite minimum degree sequence. For the general case, write $d = \gcd(r, k)$, and note that k/d divides b and n , since we assume that $k \mid rn$. Thus we can choose $0 \leq a \leq b$ such that b divides $n - ak/d$. By choosing a matching with one edge of each index in $I = \{\sum_{i \in [k]} \mathbf{u}_{i+jd} : j \in [r/d]\}$ (where addition in the subscript is modulo r) we can remove k/d vertices from each part. We can delete the edges of a vertex-disjoint such matchings from J , which only slightly weakens the conditions of the theorem, so the general case follows from the case where $b \mid n$. We also note that we did not need the full strength of the degree assumption in Theorem 2.9; for example, we could instead take the F generated by I as above, if $r > k$ is not divisible by k (so that F is connected).

Theorems 2.3 and 2.6 on matchings covering all but a constant number of vertices follow in the same way from the following common generalisation.

Theorem 7.17. *Let $1/n \ll 1/\ell \ll \gamma \ll \alpha \ll \beta, \mu \ll 1/D_F, 1/r, 1/k$. Suppose F is a (k, r) -uniform connected allocation with $|F| \leq D_F$, and \mathcal{P} partitions a set V into sets V_1, \dots, V_r each of size n . Suppose that J is a $\mathcal{P}F$ -partite k -complex on V such that*

- (i) $\delta^F(J) \geq (n, (\frac{k-1}{k} - \alpha)n, (\frac{k-2}{k} - \alpha)n, \dots, (\frac{1}{k} - \alpha)n)$,
- (ii) *for any $p \in [k-1]$ and sets $S_i \subseteq V_i$ such that $|S_i| = \lfloor pn/k \rfloor$ for each $i \in [r]$ there are at least βn^k edges of J'_k with more than p vertices in $S := \bigcup_{i \in [r]} S_i$, and*

Then J_k contains a matching which γ -represents F and covers all but at most ℓ vertices.

Proof. The proof is very similar to that of Theorem 7.11, so we just indicate the necessary modifications. Take the same hierarchy of constants as in the proof of Theorem 7.11, and also take $\psi \ll \gamma \ll 1/B, 1/C \ll \alpha$. We start by forming the same partition of $V(J)$ as in Claim 7.13 (the divisibility conditions can be ensured by discarding a constant number of vertices). Instead of Claim 7.15, the corresponding claim here is that (R', M') is (B, C) -irreducible with respect to $\mathcal{P}_{R'}$. The proof is the same, except that we apply Lemma 5.5 instead of Lemma 5.6, and there is no lattice condition to check. Next, in the analogue of Claim 7.16, we can no longer ensure property (c), but instead we maintain the property (c') $||\bigcup_{i \in [s]} V(E_i) \cap Y_j^e| - |\bigcup_{i \in [s]} V(E_i) \cap Y_{j'}^e|| \leq 4B$ for all $e \in M$ and $j, j' \in [k]$. In constructing the sets E_s , we at first cover Γ_s by e_x , $x \in \Gamma_s$ but do not attempt to balance the cluster sizes. Instead, whenever we have formed a set E_s that would cause property (c') to fail, we remedy this using suitable b -fold transferrals with $b \leq B$. To do this, write $m_s^e = \sum_{j \in [k]} |\bigcup_{i \in [s]} V(E_i) \cap Y_j^e|$ and $d_s^{ej} = |\bigcup_{i \in [s]} V(E_i) \cap Y_j^e| - m_s^e/k$ for $e \in M$, $j \in [k]$. Suppose we have formed E_s which has $d_s^{ej} > 2B$ for some $e \in M$ and $j \in [k]$. Let W be the cluster containing Y_j^e . Since each $\{e_x\}_{x \in \Gamma_s}$ uses the same number of vertices in each part V_i , $i \in [r]$, we can find another cluster W' in the same part as W , which contains some $Y_{j'}^{e'}$ for which $d_s^{e'j'} < 0$. By (B, C) -irreducibility there is a b -fold (W', W) -transferral (T, T') of

size at most C , for some $b \leq B$. To implement the transferral, we include a matching in E_s^* whose edges correspond to the edges of T , using the same procedure as described in the proof of Theorem 7.11. Similarly, if $d_s^{ej} < -2B$ for some $e \in M$ and $j \in [k]$, then we can find another cluster W' in the same part as W which contains some $Y_{j'}^{e'}$ for which $d_s^{e'j'} > 0$. We then implement a b -fold (W, W') -transferral (T, T') of size at most C , for some $b \leq B$, in the same way as before. By repeating this process while there is any $d_s^{ej} > 2B$ for some $e \in M$ and $j \in [k]$, we can construct E_s^* which when added to E_s satisfies property (c'). Thus we can greedily cover the exceptional set Γ , in such a way that for each $e \in M$ and $j, j' \in [k]$ the number of vertices used from Y_j^e and $Y_{j'}^e$ differ by at most $4B$. Now we discard at most $4B$ vertices arbitrarily from each Y_j^e to balance the cluster sizes, so that Claim 7.13 (ii) gives a perfect matching on the remaining sets. Combining all the matchings, we have covered all vertices, except for some number that is bounded by a constant independent of n . \square

8. PACKING TETRAHEDRA

In this section we prove Theorem 1.1, which determines precisely the codegree threshold for a perfect tetrahedron packing in a 3-graph G on n vertices, where $4 \mid n$ and n is sufficiently large. The theorem states that if $8 \mid n$ and $\delta(G) \geq 3n/4 - 2$, or if $8 \nmid n$ and $\delta(G) \geq 3n/4 - 1$, then G contains a perfect K_4^3 -packing. We start with a construction due to Pikhurko [34] showing that this minimum degree bound is best possible.

Proposition 8.1. *For any n divisible by 4 there exists a 3-graph G on n vertices with*

$$\delta(G) \geq \begin{cases} 3n/4 - 3 & \text{if } 8 \mid n \\ 3n/4 - 2 & \text{otherwise,} \end{cases}$$

which does not contain a perfect K_4^3 -packing.

Proof. Partition a set V of n vertices into parts V_1, V_2, V_3, V_4 such that $|V_1|$ is odd and the set sizes $|V_j|$ are as equal as possible. So $|V_j| \leq n/4 + 1$ for every $j \in [4]$, and if $8 \nmid n$ then $|V_j| = n/4$ for each n . Let G be the 3-graph on vertex set V whose edges are all $e \in \binom{V}{3}$ except for those e such that

- (i) $|e \cap V_1| = 3$,
- (ii) $|e \cap V_1| = 1$ and $|e \cap V_i| = 2$ for some $i \in \{2, 3, 4\}$, or
- (iii) $|e \cap V_2| = |e \cap V_3| = |e \cap V_4| = 1$.

To calculate $\delta(G)$, consider any pair of vertices xy . We claim that there is precisely one class V_i , $i \in [4]$ such that xyz is not an edge for $z \in V_i$. For this we just check all cases: if x, y are in the same class then $i = 1$, if $x \in V_1$ and $y \in V_j$ with $j \neq 1$ then $i = j$, and if x, y are in different parts among $\{V_2, V_3, V_4\}$ then V_i is the third of these parts. Thus $\delta(G) = n - 2 - \max\{|V_1|, |V_2|, |V_3|, |V_4|\}$, which has value $3n/4 - 3$ if $8 \mid n$ and $3n/4 - 2$ otherwise. Furthermore, we claim that any copy X of K_4^3 in G must have an even number of vertices in V_1 . To see this, note first that we cannot have $|X \cap V_1| = 3$, since there are no edges contained in V_1 . Now suppose that $|X \cap V_1| = 1$. Of the remaining 3 vertices, either 2 lie in the same part among $\{V_2, V_3, V_4\}$, or all 3 parts are represented; either way some triple in X is not an edge of G . Therefore $|X \cap V_1|$ is even. Since $|V_1|$ is odd, we conclude that G does not contain a perfect K_4^3 -packing. \square

Now we start on the proof of Theorem 1.1. We will apply our results on perfect matchings in simplicial complexes to the clique 4-complex $J_4(G)$. Note that a perfect tetrahedron packing in G is the same as a perfect matching in $J_4(G)_4$. In the first subsection, we consider the weaker assumption $\delta(G) \geq (3/4 - c)n$ for some absolute constant $c > 0$, under which we show that $J_4(G)$ satisfies the usual minimum degree sequence assumption, and a space barrier is impossible. Then Theorem 2.3 implies that G contains a tetrahedron packing that covers all but at most ℓ vertices, for some absolute constant ℓ . Thus we demonstrate that the threshold for covering all but a constant number of vertices is asymptotically different from that for covering all vertices.

The most difficult part of the proof is the analysis of divisibility barriers, which involve Turán-type problems for K_4^3 (fortunately we can solve these ones, unlike Turán's famous conjecture!) If we do not have a perfect matching in $J_4(G)_4$, then having shown that space barriers are impossible, we conclude from Theorem 2.8 that there must be a divisibility barrier. Over the second and third subsections we will show that all but one of the potential divisibility barriers are impossible, and that the possible divisibility barrier implies a stability result, that G is structurally close to the construction described in Proposition 8.1. This sets up an application of the stability method: either G is not structurally close to the construction, in which case Theorem 2.8 gives a perfect matching, or G is structurally close to the construction, in which case we give a separate argument exploiting the structure to see that again we have a perfect matching.

8.1. Packing to within a constant. As a prelude to our result on perfect tetrahedron packings, we prove the following result that demonstrates that there is a different threshold for packing to within a constant; the ingredients of its proof will also be used for the main result.

Theorem 8.2. *Suppose $1/n \ll 1/\ell \ll c \ll 1$ and G is a 3-graph on n vertices with $\delta(G) \geq (3/4 - c)n$. Then G contains a tetrahedron packing that covers all but at most ℓ vertices.*

A reformulation of the desired conclusion is that there is a matching covering all but at most ℓ vertices in J_4 , where $J = J_4(G)$ is the clique 4-complex. First we express the minimum degree sequence of J in terms of the minimum degree of G . Since J_i is complete for $i \leq 2$, we have $\delta_0(J) = n$, $\delta_1(J) = n - 1$ and $\delta_2(J) = \delta(G)$. For $\delta_3(J)$, consider any $e = xyz \in J_3 = G$. For each of the pairs xy, xz, yz , there are at most $(n - 3) - (\delta(G) - 1)$ vertices $a \in V(G) \setminus \{x, y, z\}$ that do not form an edge of G with that pair. Thus $xyza$ is a tetrahedron in G for at least $(n - 3) - 3(n - \delta(G) - 2) = 3\delta(G) - 2n + 3$ vertices a . Therefore

$$(7) \quad \delta(J_4(G)) \geq (n, n - 1, \delta(G), 3\delta(G) - 2n + 3).$$

With our current assumption $\delta(G) \geq (3/4 - c)n$, this gives $\delta(J) \geq (n, n - 1, (3/4 - c)n, (1/4 - 3c)n + 3)$, so our usual minimum degree sequence assumption is satisfied; in fact, the bounds on $\delta_1(J)$ and $\delta_2(J)$ happen to be much stronger than necessary. Next, we need to consider space barriers, of which there are 3 possible types, corresponding to sets S of size $n/4$, $n/2$ or $3n/4$. The first two cases are covered by the next proposition, which is a straightforward application of the minimum degree sequence.

Proposition 8.3. *Suppose $c \ll 1$ and G is a 3-graph on n vertices with $\delta(G) \geq (3/4 - c)n$. Then*

- (i) *for any $S \subseteq V(J)$ of size $n/4$, the number of edges of J_4 with at least 2 vertices in S is at least $n^4/3000$, i.e. J_4 is not $1/3000$ -contained in $J(S, 1)_4$, and*

(ii) for any $S \subseteq V(J)$ of size $n/2$, the number of edges of J_4 with at least 3 vertices in S is at least $n^4/2000$, i.e. J_4 is not $1/2000$ -contained in $J(S, 2)_4$.

Proof. For (i), we construct edges $v_1v_2v_3v_4$ with $v_1, v_2 \in S$. There are $n/4$ choices for v_1 , $n/4 - 1$ choices for v_2 , at least $(3/4 - c)n$ choices for v_3 , and at least $(1/4 - 3c)n$ choices for v_4 . Each such edge is counted at most 24 times, so the bound follows. Similarly, for (ii) we construct edges $v_1v_2v_3v_4$ with $v_1, v_2, v_3 \in S$. There are $n/2$ choices for v_1 , $n/2 - 1$ choices for v_2 , at least $|S| - (n - \delta(G)) \geq (1/4 - c)n$ choices for v_3 , and at least $(1/4 - 3c)n$ choices for v_4 , so again the bound follows. \square

It remains to consider the case $|S| = 3n/4$, for which we need to show that many edges of J_4 are contained in S . This follows from known bounds on the Turán density of K_4^3 and ‘supersaturation’. Here we only quote what suffices for our purposes, so we refer the reader to the survey [20] for more information. Given an r -graph F , the Turán number $\text{ex}(n, F)$ is the maximum number of edges in an F -free r -graph on n vertices. The Turán density is $\pi(F) := \lim_{n \rightarrow \infty} \binom{n}{r}^{-1} \text{ex}(n, F)$. We use the following result of Chung and Lu [4].

Theorem 8.4. $\pi(K_4^3) \leq (3 + \sqrt{17})/12 (\approx 0.5936)$.

We combine this with the following supersaturation result of Erdős and Simonovits [11].

Theorem 8.5. For any r -graph F and $a > 0$ there are $b, n_0 > 0$ so that if G is an r -graph on $n > n_0$ vertices with $e(G) > (\pi(F) + a) \binom{n}{r}$ then G contains at least $bn^{|V(F)|}$ copies of F .

Given a set $S \subseteq V(J)$ of size sn , any pair xy in S is contained in at least $\delta(G) - (1 - s)n$ edges of $G[S]$. Under the assumption $\delta(G) \geq (3/4 - c)n$, we get $|G[S]| \geq \frac{1}{3} \binom{|S|}{2} (s - 1/4 - c)n \geq (1 - 1/4s - c/s) \binom{|S|}{3}$. Note that if $s \geq 5/8$ then $1 - 1/4s \geq 3/5$, so $1 - 1/4s - c/s > \pi(K_4^3) + 1/200$ for small c . Then Theorems 8.4 and 8.5 imply the following result.

Proposition 8.6. Suppose $1/n \ll b, c \ll 1$ and G is a 3-graph on n vertices with $\delta(G) \geq (3/4 - c)n$. Then for any $S \subseteq V(J)$ with $|S| \geq 5n/8$, the number of edges of J_4 contained in S is at least bn^4 . In particular, if $|S| = 3n/4$ then J_4 is not b -contained in $J(S, 3)_4$.

Now Theorem 8.2 follows by applying Theorem 2.3 to the clique 4-complex $J_4(G)$. Indeed, the degree sequence assumption holds by (7), and there are no space barriers by Propositions 8.3 and 8.6. Thus there is a matching covering all but at most ℓ vertices in $J_4(G)_4$, as required.

8.2. Properties of index vectors. Now we need some notation and simple properties of index vectors. Let X be a set which is partitioned into r non-empty parts X_1, \dots, X_r . We let $K(X)$ denote the complete k -complex on X , where any $e \in \binom{X}{\leq k}$ is an edge of $K(X)$. Recall that the index vector $\mathbf{i}(e) \in \mathbb{Z}^r$ of a set $e \subseteq X$ has co-ordinates $i(e)_j = |e \cap X_j|$. If $\mathbf{i}(e) = \mathbf{i}$, then the possible index vectors of subsets of e are precisely those $\mathbf{i}' \in \mathbb{Z}^r$ with $\mathbf{0} \leq \mathbf{i}' \leq \mathbf{i}$, where as usual \leq is the pointwise partial order on vectors. Also recall that if J is a k -system on X , then $J_{\mathbf{i}}$ denotes the set of edges in J with index \mathbf{i} . We define the *density* of J at \mathbf{i} as

$$d_{\mathbf{i}}(J) := \frac{|J_{\mathbf{i}}|}{|K(X)_{\mathbf{i}}|}.$$

Note that this notation should not be confused with that for the relative density of a complex used in Section 6 (which we will not use in this section). Similarly to the notation used for

complexes, we let $J_{\mathbf{i}}^*$ denote the set of $e \in K(X)_{\mathbf{i}}$ such that $e' \in J$ for every strict subset $e' \subset e$. We also recall that we write $|\mathbf{i}| = \sum_{j \in [r]} i_j$, and that \mathbf{u}_i denotes the standard basis vector with i th coordinate 1 and all other coordinates 0. Given an index vector $\mathbf{i} = (i_1, \dots, i_r)$, we write $\partial \mathbf{i}$ for the multiset that contains i_j copies of $\mathbf{i} - \mathbf{u}_j$ for each $j \in [r]$. The next proposition sets out some useful properties linking the densities of different indices.

Proposition 8.7. *Let J be a k -complex on a set X partitioned as (X_1, \dots, X_r) . Then*

- (i) *for any \mathbf{i}, \mathbf{i}' with $\mathbf{i}' \leq \mathbf{i}$ we have $d_{\mathbf{i}'}(J) \geq d_{\mathbf{i}}(J)$,*
- (ii) *for any \mathbf{i} we have $\sum_{j \in [r]} d_{\mathbf{i} + \mathbf{u}_j}(J) |X_j| \geq \delta_{|\mathbf{i}|}(J) d_{\mathbf{i}}(J)$,*
- (iii) *if $J_{\mathbf{i}} = J_{\mathbf{i}}^*$ for some $\mathbf{i} = (i_1, \dots, i_r)$, then $d_{\mathbf{i}}(J) \geq 1 - \sum_{j \in [r]} i_j(1 - d_{\mathbf{i} - \mathbf{u}_j}(J))$, and*
- (iv) *for any \mathbf{i} we have $\prod_{\mathbf{i}' \in \partial \mathbf{i}} d_{\mathbf{i}'}(J) \geq d_{\mathbf{i}}(J)^{|\mathbf{i}| - 1} + O(1/|X|)$.*

Proof. For (i), it is sufficient to consider the case when $|\mathbf{i}'| = |\mathbf{i}| - 1$. We can write $\mathbf{i}' = \mathbf{i} - \mathbf{u}_j$ for some $j \in [r]$ such that $i_j \geq 1$. The required density inequality is a variant of the Local LYM inequality (see e.g. [2, Theorem 3.3]). We briefly give the standard double-counting argument, which is as follows. Consider the pairs (e', e) with $e' \in J_{\mathbf{i}'}$, $e \in J_{\mathbf{i}}$ and $e' \subseteq e$. The number of such pairs is at least $i_j |J_{\mathbf{i}}|$, and at most $(|X_j| - i_j + 1) |J_{\mathbf{i}'}|$. It follows that $|J_{\mathbf{i}'}|/|J_{\mathbf{i}}| \geq i_j / (|X_j| - i_j + 1) = \binom{|X_j|}{i_j - 1} / \binom{|X_j|}{i_j} = |K(X)_{\mathbf{i}'}|/|K(X)_{\mathbf{i}}|$, which gives (i).

For (ii), we consider the pairs (e, v) with $e \in J_{\mathbf{i}}$ and $e \cup \{v\} \in J$. The number of such pairs is at least $\delta_{|\mathbf{i}|}(J) |J_{\mathbf{i}}|$, and at most $\sum_{j \in [r]} (i_j + 1) |J_{\mathbf{i} + \mathbf{u}_j}|$. This gives the stated inequality, using $|J_{\mathbf{i}}| = d_{\mathbf{i}}(J) |K(X)_{\mathbf{i}}|$, $|J_{\mathbf{i} + \mathbf{u}_j}| = d_{\mathbf{i} + \mathbf{u}_j}(J) |K(X)_{\mathbf{i} + \mathbf{u}_j}|$, and $|X_j| |K(X)_{\mathbf{i}}| = (i_j + 1) |K(X)_{\mathbf{i} + \mathbf{u}_j}|$.

For (iii), choose e in $K(X)_{\mathbf{i}}$ uniformly at random. For each $e' \subset e$ with $|e'| = |e| - 1$, writing $\mathbf{i}(e') = \mathbf{i} - \mathbf{u}_j$, the probability that e' is *not* an edge of J is $1 - d_{\mathbf{i} - \mathbf{u}_j}(J)$. Thus the probability that every strict subset $e' \subset e$ is an edge of J is at least $1 - \sum_{j \in [r]} i_j(1 - d_{\mathbf{i} - \mathbf{u}_j}(J))$. Since $J_{\mathbf{i}} = J_{\mathbf{i}}^*$, this event implies that $e \in J$, so we have the stated density.

For (iv), we first consider the multipartite case, when each i_j is 0 or 1, and let $I = \{j : i_j = 1\}$. We apply Shearer's Lemma (see [3]), which states that if \mathcal{A} and \mathcal{F} are families of subsets of S such that each element of S is contained in at least ℓ elements of \mathcal{A} , and $\mathcal{F}_A = \{A \cap F : F \in \mathcal{F}\}$ for $A \in \mathcal{A}$, then $\prod_{A \in \mathcal{A}} |\mathcal{F}_A| \geq |\mathcal{F}|^{\ell}$. Here we take $S = X_I$, $\mathcal{F} = J_{\mathbf{i}}$, and $\mathcal{A} = \{X_{I \setminus j}\}_{j \in I}$, so $\ell = |\mathbf{i}| - 1$. Then $\prod_{j \in I} |J_{\mathbf{i} - \mathbf{u}_j}| \geq |J_{\mathbf{i}}|^{|\mathbf{i}| - 1}$, which gives the result (even without the $O(1/|X|)$ term). For the general case we apply the multipartite case to an auxiliary k -partite k -complex H , whose parts $X'_1, \dots, X'_{|\mathbf{i}|}$ consist of i_j copies of X_j for $j \in [r]$, and whose edges are all k -partite copies of each edge in $\bigcup_{\mathbf{i}' \leq \mathbf{i}} J_{\mathbf{i}'}$ (so, for example, each edge of $J_{\mathbf{i}}$ corresponds to $\prod_{j \in [r]} i_j!$ of H_k). Then $d_{\mathbf{1}}(H) = d_{\mathbf{i}}(J) + O(1/|X|)$ and $\{d_{\mathbf{1} - \mathbf{u}_j}(H)\}_{j \in [|\mathbf{i}|]}$ has i_j copies of $d_{\mathbf{i} - \mathbf{u}_j}(J) + O(1/|X|)$ for $j \in [r]$, so the result follows. \square

8.3. Divisibility barriers with two parts. For the perfect packing result, we need to consider divisibility barriers. We start with the case of a bipartition, for which there are two natural candidates for a barrier: (i) almost all tetrahedra are ‘even’, i.e. have an even number of vertices in both parts, and (ii) almost all tetrahedra are ‘odd’, i.e. have an odd number of vertices in both parts. We start by showing that these are indeed the only two possibilities. Note that here, and for most of the proof, we only assume the weaker degree hypothesis used in the previous subsection; the exact degree condition is only used in Lemma 8.10.

Lemma 8.8. *Suppose $1/n \ll \mu, c \ll 1$ and G is a 3-graph on n vertices with $\delta(G) \geq (3/4 - c)n$. Write $J = J_4(G)$ for the clique 4-complex of G . Suppose \mathcal{P} is a partition of V*

into two parts V_1, V_2 of size at least $n/4 - \mu n$ such that $L_P^\mu(J_4)$ is incomplete. Then either (i) there are fewer than $3\mu n^4$ odd tetrahedra, or (ii) there are fewer than $3\mu n^4$ even tetrahedra.

Proof. Suppose for a contradiction that there are at least $3\mu n^4$ odd tetrahedra and at least $3\mu n^4$ even tetrahedra. Then $L := L_P^\mu(J_4)$ contains at least one of $(1, 3)$ and $(3, 1)$, and also at least one of $(4, 0)$, $(2, 2)$ and $(0, 4)$. By (7), we have $\delta(J) \geq (n, n-1, (3/4-c)n, (1/4-3c)n)$. Next we apply Proposition 8.7(ii) to $(1, 1)$, which gives $d_{(2,1)}(J)|V_1| + d_{(1,2)}(J)|V_2| \geq (3/4-c)n$, and to $(2, 0)$, which gives $d_{(3,0)}(J)|V_1| + d_{(2,1)}(J)|V_2| \geq (3/4-c)n$. Summing the inequalities we deduce $d_{(2,1)}(J)n \geq (3/2-2c)n - |V_2| - |V_1| = (1/2-2c)n$. Now applying Proposition 8.7(ii) to $(2, 1)$ we get $d_{(3,1)}(J)|V_1| + d_{(2,2)}(J)|V_2| \geq (1/2-2c)(1/4-3c)n$, so L contains at least one of $(3, 1)$ and $(2, 2)$. Similarly, L contains at least one of $(1, 3)$ and $(2, 2)$. Now we can deduce that $(-1, 1) \in L$. Indeed, if $(2, 2) \in L$ this holds since $(-1, 1) = (1, 3) - (2, 2) = (2, 2) - (3, 1)$ and L contains at least one of $(1, 3)$ and $(3, 1)$. On the other hand, if $(2, 2) \notin L$ then L contains both $(3, 1)$ and $(1, 3)$. Also, L contains at least one of $(4, 0)$ and $(0, 4)$. Since $(-1, 1) = (3, 1) - (4, 0) = (0, 4) - (1, 3)$, again we get $(-1, 1) \in L$. However, this contradicts the assumption that L is incomplete. \square

Next we show that it is impossible to avoid even tetrahedra.

Lemma 8.9. *Suppose $1/n \ll b, c \ll 1$ and G is a 3-graph on n vertices with $\delta(G) \geq (3/4-c)n$. Then for any bipartition of $V(G)$, there are at least bn^4 copies of K_4^3 with an even number of vertices in both parts.*

Proof. Consider any partition (A, B) of $V(G)$. Form a digraph D on vertex set $V(D) := \binom{A}{2} \cup \binom{B}{2}$, where there is an edge from $P \in V(D)$ to $Q \in V(D)$ precisely if $P \cup \{q\}$ is an edge of G for each $q \in Q$. Then there is a one-to-one correspondence between 2-cycles $\{PQ, QP\}$ in D and copies of K_4^3 in G with an even number of vertices in A . We write $|A| = an$, where without loss of generality we have $0 \leq a \leq 1/2$. We can also assume that $a \geq 3/8$, as otherwise $|B| \geq 5n/8$, so the desired conclusion follows from Proposition 8.6. Then by convexity,

$$|V(D)| = \binom{|A|}{2} + \binom{|B|}{2} \leq \binom{3n/8}{2} + \binom{5n/8}{2} \leq \frac{17n^2}{64}.$$

Also, for any $P \in V(D)$, we can write $d_D^+(P) = \binom{d_A(P)}{2} + \binom{d_B(P)}{2}$, where $d_X(P)$ is the number of vertices $x \in X$ such that $P \cup \{x\} \in G$, for $X \in \{A, B\}$. Since $d_A(P) + d_B(P) = \delta(G) \geq (3/4-c)n$, by convexity

$$\delta^+(D) \geq 2 \binom{(3/4-c)n/2}{2} > \frac{9n^2}{64} - cn^2.$$

So the number of 2-cycles in D is at least $(9|V(D)|/17 - cn^2)|V(D)| - (|V(D)|^2) \geq bn^4$, as required. \square

Now we consider odd tetrahedra. Note that the construction in Proposition 8.1 gives a 3-graph G on n vertices partitioned into two parts V_1 and $V_2 \cup V_3 \cup V_4$, so that $|V_1|$ is an odd number close to $n/4$, and all tetrahedra are even with respect to the bipartition. The next lemma shows that the minimum degree condition of Theorem 1.1 is sufficient to prevent any such partition in G ; this is the only part of the proof of Theorem 1.1 for which the minimum degree bound is tight.

Lemma 8.10. *Suppose G is a 3-graph on n vertices with $4 \mid n$ and $\delta(G) \geq 3n/4 - 1$. Let (A, B) be a partition of $V(G)$ with $3n/16 < |A| < 5n/16$. Then G contains a copy of K_4^3 with an odd number of vertices in both parts.*

Proof. Consider such a partition (A, B) , and suppose for a contradiction that there is no odd tetrahedron. By (7), for any edge $e \in G$, the number of vertices v such that $e \cup \{v\}$ induces a copy of K_4^3 is at least $3\delta(G) - 2n + 3 \geq n/4$. If e has an odd number of vertices in A , then each such vertex v must also lie in A . Consider any $u \in A$ and $v \in B$. Since $\delta(G) > n - |B|$ we may choose $v' \in B$ such that $e = uvv'$ is an edge of G . Then e has an odd number of vertices in A , so there must be at least $n/4$ vertices $u' \in A$ such that $e \cup \{u'\}$ induces a copy of K_4^3 . Each such u' is distinct from u , so we must have $|A| \geq n/4 + 1$.

Now consider any distinct $x, y \in B$. There are at least $\delta(G) \geq 3n/4 - 1$ vertices $z \in V(G)$ such that $xyz \in G$. At most $|B| - 2 \leq 3n/4 - 3$ of these lie in B , so we may choose some such $z \in A$. Then $e' = xyz$ has an odd number of vertices in A , so again there must be at least $n/4$ vertices $w \in A$ such that $e' \cup \{w\}$ induces a copy of K_4^3 in G . But x and y were arbitrary, so we deduce that for any $x, y \in B$, there are at most $|A| - n/4 < n/16$ vertices $z \in A$ for which xyz is not an edge of G . Since $|A| > 3n/16$, for any edge $x'y'z'$ in $G[B]$, there is some $u' \in A$ such that $u'x'y'z'$ induces a copy of K_4^3 in G , which contradicts the assumption that there is no odd tetrahedron. \square

The previous lemma will be useful when considering the extremal case, but a single odd tetrahedron does not suffice to rule out a divisibility barrier. To find many odd tetrahedra we need a result relating the triangle density to the edge density in any graph. Given a graph G on n vertices, the edge density of G is $d(G) = e(G) \binom{n}{2}^{-1}$, and the triangle density is $\Delta(G) = N_3(G) \binom{n}{3}^{-1}$, where $N_3(G)$ is the number of triangles in G . In the following theorem, the first part follows from a result of Goodman [14], and the second from a result of Lovász and Simonovits [28, Theorem 2]. (Razborov [35] proved an asymptotically tight general result.)

Theorem 8.11. *Suppose G is a graph on n vertices. Then*

- (i) $\Delta(G) \geq d(G)(2d(G) - 1)$, and
- (ii) *if $1/n \ll \varepsilon \ll \varepsilon' \ll 1$, $d(G) = 2/3 \pm \varepsilon$ and $\Delta(G) = 2/9 \pm \varepsilon$, then we may partition $V(G)$ into sets V_1, V_2, V_3 each of size at least $n/3 - \varepsilon'n$ so that $d(G[V_i]) \leq \varepsilon'$ for each $i \in [3]$.*

Now we can prove the following result, which states that, even under the weaker degree assumption $\delta(G) \geq (3/4 - c)n$, either there is no divisibility barrier for odd tetrahedra, or G is close to the extremal example. Let \mathcal{P} partition a set V into parts V_1, V_2, V_3, V_4 . We say that an index vector \mathbf{i} with respect to \mathcal{P} is *bad* if it is not used by edges of the extremal example, i.e. \mathbf{i} is one of $(3, 0, 0, 0)$, $(1, 2, 0, 0)$, $(1, 0, 2, 0)$, $(1, 0, 0, 2)$ or $(0, 1, 1, 1)$; otherwise \mathbf{i} is *good*.

Lemma 8.12. *Let $1/n \ll b, c \ll \gamma \ll 1$ and G be a 3-graph on n vertices with $\delta(G) \geq (3/4 - c)n$. Suppose that there is some partition of $V(G)$ into parts U_1 and U_2 each of size at least $n/4 - bn$ such that at most bn^4 copies of K_4^3 in G are ‘odd’, i.e. have an odd number of vertices in each part. Then there is a partition of $V(G)$ into parts V_1, V_2, V_3, V_4 each of size at least $n/4 - \gamma n$, such that V_1 is equal to one of U_1 or U_2 , and $d_{\mathbf{i}}(G) \geq 1 - \gamma$ for each good \mathbf{i} .*

Proof. Introduce new constants with $b, c \ll \alpha \ll \beta \ll \beta' \ll \gamma$. Let $J = J_4(G)$ be the clique 4-complex of G . By (7), we have $\delta(J) \geq (n, n-1, (3/4-c)n, (1/4-3c)n)$. Write

$$\begin{aligned}\tau &:= d_{(3,0)}(J) = d(G[U_1]), & \tau' &:= d_{(0,3)}(J) = d(G[U_2]) \\ \rho &:= d_{(2,1)}(J) - \alpha, & \rho' &:= d_{(1,2)}(J) - \alpha.\end{aligned}$$

Let Z be the set of vertices that lie in more than \sqrt{bn}^3 odd tetrahedra. Then $|Z| < 4\sqrt{bn}$. Let $U'_1 = \{v \in U_1 : d(G(v)[U_2]) \geq \rho'\}$. Then $\rho' + \alpha = d_{(1,2)}(J) = |U_1|^{-1} \sum_{v \in U_1} d(G(v)[U_2]) \leq |U_1|^{-1}(|U'_1| + |U_1 \setminus U'_1|\rho')$, so $|U'_1| \geq \alpha|U_1|$. Thus we can choose $v \in U'_1 \setminus Z$. We claim that $G(v)[U_2]$ must have triangle density at most $1 - \tau' + \alpha$. Otherwise, we would have at least $\alpha \binom{|U_2|}{3} > \sqrt{bn}^3$ triples in U_2 that are both triangles in $G(v)[U_2]$ and edges of $G[U_2]$. But these triples form odd tetrahedra with v , which contradicts $v \notin Z$. So $G(v)[U_2]$ has triangle density at most $1 - \tau' + \alpha$, and, since $v \in U'_1$, edge density at least ρ' . Now Theorem 8.11(i) gives

$$(8) \quad \rho'(2\rho' - 1) + \tau' - \alpha \leq 1.$$

The same argument with the roles of U_1 and U_2 switched shows that

$$(9) \quad \rho(2\rho - 1) + \tau - \alpha \leq 1.$$

Next, write $a = |U_1|/n$, where without loss of generality

$$(10) \quad 1/4 - b \leq a \leq 1/2.$$

Since $d_i(J) = 1$ for any \mathbf{i} with $|\mathbf{i}| = 2$, by Proposition 8.7(ii) applied to each such \mathbf{i} we have

$$(11) \quad \tau a + (\rho + \alpha)(1 - a) \geq 3/4 - c,$$

$$(12) \quad \tau'(1 - a) + (\rho' + \alpha)a \geq 3/4 - c,$$

$$(13) \quad (\rho + \alpha)a + (\rho' + \alpha)(1 - a) \geq 3/4 - c.$$

We claim that inequalities (8)-(13) imply the estimates

$$(14) \quad a = 1/4 \pm \beta, \quad \tau \leq \beta, \quad \rho \geq 1 - \beta, \quad \rho' = 2/3 \pm \beta, \quad \tau' = 7/9 \pm \beta.$$

The proof of this claim requires some calculations, which are tiresome, but not difficult. We first consider inequality (11). The left hand side is linear in a , so its maximum is attained at (at least) one of the extreme values $a = 1/2$ and $a = 1/4 - b$. Thus we either have (A) $3/4 - c \leq \tau/2 + (\rho + \alpha)/2$ or (B) $3/4 - c \leq (1/4 - b)\tau + (\rho + \alpha)(3/4 + b)$. We will show that the first case (A) leads to a contradiction. For it implies

$$\begin{aligned}1 - 2c &= 2(3/4 - c) - 1/2 \leq \tau + \rho + \alpha - 1/2 = \tau + \rho(2\rho - 1) - (2\rho - 1)^2/2 + \alpha \\ &\leq 1 - (2\rho - 1)^2/2 + 2\alpha\end{aligned}$$

by (9), so $(2\rho - 1)^2 \leq 5\alpha$. Then $\rho = 1/2 \pm \sqrt{5\alpha}$, so (11) gives

$$3/4 - c \leq a + (1/2 + \sqrt{6\alpha})(1 - a) = 1/2 + \sqrt{6\alpha} + (1/2 - \sqrt{6\alpha})a,$$

so $a \geq 1/2 - 9\sqrt{\alpha}$. Since $a \leq 1/2$, we can write $a = 1/2 \pm 9\sqrt{\alpha}$, and apply the same reasoning to inequality (12). This gives $3/4 - c \leq (\tau' + \rho' + \alpha)(1/2 + 9\sqrt{\alpha})$, so

$$\begin{aligned}1 - 2c &\leq (\tau' + \rho' + \alpha)(1 + 18\sqrt{\alpha}) - 1/2 \leq \tau' + \rho'(2\rho' - 1) - (2\rho' - 1)^2/2 + 38\sqrt{\alpha} \\ &\leq 1 - (2\rho' - 1)^2/2 + 39\sqrt{\alpha},\end{aligned}$$

by (8). Then $(2\rho' - 1)^2 \leq 80\sqrt{\alpha}$, so $\rho' = 1/2 \pm 10\alpha^{1/4}$. But now substituting a, ρ, ρ' in (13) gives a contradiction. Thus case (A) is impossible, so we must have case (B). This implies

$$\begin{aligned} 1 - 4c &= 4(3/4 - c) - 2 \leq \tau + 3\rho + 4\alpha - 2 = \tau + \rho(2\rho - 1) - 2(\rho - 1)^2 + 4\alpha \\ &\leq 1 - 2(\rho - 1)^2 + 5\alpha \end{aligned}$$

by (9). Then $(\rho - 1)^2 \leq 3\alpha$, so $\rho \geq 1 - \sqrt{3\alpha}$. Then (9) gives

$$\tau \leq 1 + \alpha - (1 - \sqrt{3\alpha})(1 - 2\sqrt{3\alpha}) < 6\sqrt{\alpha}.$$

Now (11) gives $3/4 - c \leq 6\sqrt{\alpha} + (1 + \alpha)(1 - a)$, so

$$a < 1/4 + 7\sqrt{\alpha}.$$

Then (13) gives $3/4 - c \leq (1 + \alpha)(1/4 + 7\sqrt{\alpha}) + (\rho' + \alpha)(3/4 + b)$, so

$$\rho' \geq 2/3 - 10\sqrt{\alpha}.$$

Now (8) implies

$$\tau' \leq 1 + \alpha - (2/3 - 10\sqrt{\alpha})(1/3 - 20\sqrt{\alpha}) < 7/9 + 20\sqrt{\alpha}.$$

Next, since $\tau' \leq 1 + \alpha - \rho'(2\rho' - 1)$ by (8), substituting in (12) gives

$$\begin{aligned} 3/4 - c &\leq (1 + \alpha - \rho'(2\rho' - 1))(3/4 + b) + (\rho' + \alpha)(1/4 + 7\sqrt{\alpha}) \\ &\leq 3/4 - \frac{3}{2}\rho'(\rho' - 2/3) + 8\sqrt{\alpha}, \end{aligned}$$

so $\rho' \leq 2/3 + 9\sqrt{\alpha}$. Finally, substituting this in (12) gives

$$3/4 - c \leq \tau'(3/4 + b) + (2/3 + 10\sqrt{\alpha})(1/4 + 7\sqrt{\alpha}),$$

so $\tau' \geq 7/9 - 10\sqrt{\alpha}$. Thus we have verified all the estimates in (14).

Applying (14) to the vertex $v \in U'_1 \setminus Z$ chosen above, we see that the graph $G(v)[U_2]$ has edge density at least $2/3 - \beta$ and triangle density at most $2/9 + 2\beta$. So by Theorem 8.11(ii) we may partition U_2 into sets V_2, V_3 and V_4 each of size at least $n/4 - \gamma n$ such that for each $i \in \{2, 3, 4\}$ we have $d(G(v)[V_i]) \leq \beta'$. By Theorem 8.11(i), the triangle density of $G(v)[U_2]$ is at least $\rho'(2\rho' - 1) \geq 2/9 - 2\beta$. Since $d(G[U_2]) = \tau' \geq 7/9 - \beta$, we see that all but at most $4\beta \binom{|U_2|}{3}$ triples in U_2 either form triangles in $G(v)[U_2]$ or edges in $G[U_2]$; otherwise, we would have at least $\beta \binom{|U_2|}{3} > \sqrt{b}n^3$ odd tetrahedra containing v , contradicting $v \notin Z$. Now write $V_1 := U_1$ and consider index vectors with respect to the partition (V_1, V_2, V_3, V_4) . Then for each $i \in \{2, 3, 4\}$, since $d(G(v)[V_i]) \leq \beta'$ we have $d_{3\mathbf{u}_i}(J) = d(G[V_i]) \geq 1 - \gamma$. Also, since all but at most $\beta'n^3$ triples of index $(0, 1, 1, 1)$ are triangles of $G(v)$ we have $d_{(0,1,1,1)}(J) < 100\beta'$. Applying Proposition 8.7(ii) to $\mathbf{i} = (2, 0, 0, 0)$ we have

$$d_{(3,0,0,0)}(J)|V_1| + d_{(2,1,0,0)}(J)|V_2| + d_{(2,0,1,0)}(J)|V_3| + d_{(2,0,0,1)}(J)|V_4| \geq (3/4 - c)n.$$

Since $d_{(3,0,0,0)}(J) = d(G[V_1]) = \tau \leq \beta$ and $|V_1| = (1/4 \pm \beta)n$, each of $d_{(2,1,0,0)}(J), d_{(2,0,1,0)}(J), d_{(2,0,0,1)}(J)$ must be at least $1 - \gamma$. Similarly, since $d_{(0,1,1,1)}(J) < 100\beta'$, applying Proposition 8.7(ii) to each of $(0, 1, 1, 0), (0, 1, 0, 1)$ and $(0, 0, 1, 1)$ we see that $d_{\mathbf{i}}(J) \geq 1 - \gamma$ for all remaining good index vectors. \square

8.4. Divisibility barriers with more parts. In this subsection we show that, if there are no divisibility barriers with two parts, then there are no divisibility barriers with more than two parts. We consider separately the cases of three parts and four parts (there cannot be more because of the minimum size of the parts). We will repeatedly use the following observation, which is immediate from Proposition 8.7(ii).

Proposition 8.13. *Let $\mu \ll d$ and \mathcal{P} partition a set X of n vertices into r parts. Suppose J is a k -complex on X with $d_{\mathbf{i}}(J) \geq d$ and $\delta_{|\mathbf{i}|}(J) \geq dn$ for some index vector $\mathbf{i} \in \mathbb{Z}^r$. Then there is some $j \in [r]$ such that $\mathbf{i} + \mathbf{u}_j \in L_{\mathcal{P}}^{\mu}(J_{|\mathbf{i}|+1})$, i.e. there are at least $\mu n^{|\mathbf{i}|+1}$ edges in J with index vector $\mathbf{i} + \mathbf{u}_j$.*

We start by considering divisibility barriers with three parts. Recall that a lattice $L \subseteq \mathbb{Z}^d$ is minimal if it does not contain any difference $\mathbf{u}_i - \mathbf{u}_j$ of distinct unit vectors $\mathbf{u}_i, \mathbf{u}_j$.

Lemma 8.14. *Let $1/n \ll c \ll \mu \ll 1$, and G be a 3-graph on n vertices with $\delta(G) \geq (3/4 - c)n$. Write $J = J_4(G)$ for the clique 4-complex of G . Then there is no partition \mathcal{P} of $V(G)$ into three parts of size at least $n/4 - \mu n$ such that $L_{\mathcal{P}}^{\mu}(J_4)$ is minimal and incomplete.*

Proof. Introduce constants α, β, β' with $\mu \ll \alpha \ll \beta \ll \beta' \ll 1$. We need the following claim.

Claim 8.15. *Suppose (U_1, U_2) is a partition of V with $|U_1| = an$ and $1/4 - \alpha \leq a \leq 1/2 + \alpha$. Then at least one of the following holds:*

- (i) *there are at least μn^4 tetrahedra with 3 vertices in U_1 ,*
- (ii) *$a = 1/4 \pm \beta$ and $d_{(2,1)}(J) > 1 - \beta'$,*
- (iii) *$a = 1/2 \pm \beta$ and $d_{(1,2)}(J) > 1 - \beta'$.*

To prove the claim, suppose option (i) does not hold, i.e. there are at most μn^4 tetrahedra with 3 vertices in U_1 . We repeat the first part of the proof of Lemma 8.12 with μ in place of b . Define τ, τ', ρ, ρ' as in that proof. Then there is a vertex $v \in U_2$ with $d(G(v)[U_1]) \geq \rho$ that belongs to at most $\sqrt{\mu} n^3$ odd tetrahedra (interchanging the roles of U_1 and U_2 from before). It follows that $G(v)[U_1]$ has triangle density at most $1 - \tau + \alpha$, so (9) holds. Also (11) and (13) hold as before.

First we show that $1/4 + \beta < a < 1/2 - \beta$ leads to a contradiction. Since (11) is linear in a , we either have (A) $3/4 - c \leq (1/4 + \beta)\tau + (3/4 - \beta)(\rho + \alpha)$ or (B) $3/4 - c \leq (1/2 - \beta)\tau + (1/2 + \beta)(\rho + \alpha)$. Consider case (A). It implies $1 - 4c = 4(3/4 - c) - 2 \leq \tau + 3\rho - 2 - 4\beta(\rho - \tau - \alpha/\beta) - \alpha = \tau - \alpha + \rho(2\rho - 1) - 2(\rho - 1)^2 - 4\beta(\rho - \tau - \alpha/\beta) \leq 1 - 2(\rho - 1)^2 - 4\beta(\rho - \tau - \alpha/\beta)$, so $2(\rho - 1)^2 \leq 4c - 4\beta(\rho - \tau - \alpha/\beta)$. However, this implies $\rho > 1 - 2\sqrt{\beta}$, so $\tau < 10\sqrt{\beta}$ by (9), which contradicts $4c - 4\beta(\rho - \tau - \alpha/\beta) \geq 0$. Now consider case (B). It implies $1 - 2c = 2(3/4 - c) - 1/2 \leq \tau + \rho - 1/2 - 2\beta(\tau - \rho - \alpha/\beta) = \tau + \rho(2\rho - 1) - (2\rho - 1)^2/2 - 2\beta(\tau - \rho - \alpha/\beta) \leq 1 + \alpha - (2\rho - 1)^2/2 - 2\beta(\tau - \rho - \alpha/\beta)$, so $(2\rho - 1)^2 + 4\beta(\tau - \rho - \alpha/\beta) \leq 4c + 2\alpha$. Then $\rho = 1/2 \pm 3\sqrt{\beta}$ and $\beta(\tau - \rho - \alpha/\beta) \leq c + \alpha/2$. Now (11) gives $3/4 - c \leq \tau(1/2 - \beta) + (1/2 + 3\sqrt{\beta} + \alpha)(3/4 - \beta)$, so $\tau \geq 3/4$. But now $\beta/5 \leq \beta(\tau - \rho - \alpha/\beta) \leq c + \alpha/2$, which is a contradiction.

Next suppose that $a = 1/4 \pm \beta$. By (11) we have $3/4 - c \leq (1/4 + \beta)\tau + (3/4 + \beta)(\rho + \alpha)$. Similarly to case (A) we have $1 - 4c = 4(3/4 - c) - 2 \leq \tau + 3\rho - 2 + 9\beta \leq 1 - 2(\rho - 1)^2 + \alpha + 9\beta$, so $d_{(2,1)}(J) = \rho > 1 - \beta'$, which is option (ii). Finally, suppose that $a = 1/2 \pm \beta$. By (11) we have $3/4 - c \leq (1/2 + \beta)(\tau + \rho + \alpha)$. Similarly to case (B) we have $1 - 2c = 2(3/4 - c) - 1/2 \leq \tau + \rho - 1/2 + 5\beta \leq 1 + \alpha - (2\rho - 1)^2/2 + 5\beta$, so $\rho = 1/2 \pm 2\sqrt{\beta}$. Now (13)

gives $3/4 - c \leq (1/2 + \beta)(\rho + \alpha + \rho' + \alpha)$, so $d_{(1,2)}(J) = \rho' > 1 - \beta'$, which is option (iii). This proves Claim 8.15.

Returning to the proof of the lemma, suppose for a contradiction we have a partition \mathcal{P} of V into parts V_1, V_2, V_3 of size at least $n/4 - \mu n$ such that $L_{\mathcal{P}}^{\mu}(J_4)$ is minimal and incomplete. Without loss of generality $|V_1| \geq n/3$. Also, since each of V_1, V_2, V_3 has size at least $n/4 - \mu n$, each has size at most $n/2 + 2\mu n$. Now recall that $\delta(J) \geq (n, n-1, (3/4 - c)n, (1/4 - 3c)n)$ by (7). Since J_2 is complete, for any \mathbf{i} with $|\mathbf{i}| = 2$, Proposition 8.7(ii) gives $(3/4 - c)n \leq \sum_{j \in [3]} d_{\mathbf{i} + \mathbf{u}_j}(J)|V_j| \leq d_{\mathbf{i} + \mathbf{u}_1}(J)|V_1| + n - |V_1|$. This gives $(1/4 + c)n \geq (1 - d_{\mathbf{i} + \mathbf{u}_1}(J))|V_1| \geq (1 - d_{\mathbf{i} + \mathbf{u}_1}(J))n/3$, so $d_{\mathbf{i} + \mathbf{u}_1}(J) \geq 1/5$. Thus $d_{\mathbf{i}}(J) \geq 1/5$ for every $\mathbf{i} = (i_1, i_2, i_3)$ with $|\mathbf{i}| = 3$ and $i_1 \geq 1$. For each such \mathbf{i} , Proposition 8.13 gives some $j \in [3]$ with $\mathbf{i} + \mathbf{u}_j \in L := L_{\mathcal{P}}^{\mu}(J_4)$. In particular, there is some $j \in [3]$ such that $(1, 1, 1) + \mathbf{u}_j \in L$. Without loss of generality $j = 1$ or $j = 2$, as if $j = 3$ we can rename V_2 and V_3 to get $j = 2$.

Suppose first that $j = 1$, i.e. $(2, 1, 1) \in L$. We apply Claim 8.15 to the partition with $U_1 = V_1$ and $U_2 = V_2 \cup V_3$. Option (i) cannot hold, as then either $(3, 1, 0) \in L$, so $(3, 1, 0) - (2, 1, 1) = (1, 0, -1) \in L$, or $(3, 0, 1) \in L$, so $(3, 0, 1) - (2, 1, 1) = (1, -1, 0) \in L$, which contradicts minimality of L . Also, option (ii) cannot hold, since $|V_1| \geq n/3$. Thus option (iii) holds, i.e. $|U_1| = an$ with $a = 1/2 \pm \beta$ and $d_{(1,2)}(J) > 1 - \beta'$ with respect to (U_1, U_2) . It follows that $d_{\mathbf{i}}(J) > 1 - 10\beta'$ with respect to \mathcal{P} for any $\mathbf{i} = (i_1, i_2, i_3)$ with $i_1 = 1$, $i_2 + i_3 = 2$ and $i_2, i_3 \geq 0$. Now Proposition 8.7(ii) applied to $(0, 1, 1)$ gives $(3/4 - c)n \leq (1/2 + \beta)n + (d_{(0,2,1)}(J) + d_{(0,1,2)}(J))(1/4 + 2\beta)n$, so without loss of generality $d_{(0,2,1)}(J) > 1/3$. Since $J = J_4(G)$ is a clique complex, we have $J_{(1,2,1)} = J_{(1,2,1)}^*$, so we can apply Proposition 8.7(iii) to get $d_{(1,2,1)}(J) \geq 1 - 30\beta' - 2/3 \geq 1/4$. Thus $(1, 2, 1) \in L$, so $(1, 2, 1) - (2, 1, 1) = (-1, 1, 0) \in L$, contradicting minimality.

Now suppose that $j = 2$, i.e. $(1, 2, 1) \in L$. We apply Claim 8.15 to the partition with $U_1 = V_2$ and $U_2 = V_1 \cup V_3$. As in the case $j = 1$, option (i) cannot hold by minimality of L . Also, option (iii) cannot hold, since $|V_1| \geq n/3$ and $|V_3| \geq n/4 - \mu n$. Thus option (ii) holds, i.e. $|U_2| = an$ with $a = 1/4 \pm \beta$ and $d_{(2,1)}(J) > 1 - \beta'$ with respect to (U_1, U_2) . It follows that $d_{(1,2,0)}(J)$ and $d_{(0,2,1)}(J)$ are both at least $1 - 10\beta'$ (say) with respect to \mathcal{P} . Next note that $(2, 2, 0) \notin L$ and $(2, 1, 1) \notin L$ by minimality of L . Thus $d_{(2,2,0)}(J)$ and $d_{(2,1,1)}(J)$ are both at most 500μ (say). Applying Proposition 8.7(ii) to $(2, 1, 0)$ gives $(1/4 - 3c)n \leq d_{(3,1,0)}(J)|V_1| + 1000\mu n$, so $d_{(3,1,0)}(J) \geq 1/3$ (say). Then Proposition 8.7(i) gives $d_{(2,1,0)}(J) \geq 1/3$. Now choose $e = xx'yy'$ in $K(V)_{(2,2,0)}$ uniformly at random, with $x, x' \in V_1$ and $y, y' \in V_2$. Given xx' , let $\rho_{xx'}|V_2|$ be the number of $v_2 \in V_2$ such that $xx'v_2 \in G$. Then $\mathbb{E}_{xx'}\rho_{xx'} = d_{(2,1,0)}(J) \geq 1/3$. So the probability that $xx'y$ and $xx'y'$ are both edges is at least $\mathbb{E}_{xx'}\rho_{xx'}(\rho_{xx'} - 5/n) \geq 1/10$, by Cauchy-Schwartz. Also, each of xyy' and $x'yy'$ is an edge with probability at least $d_{(1,2,0)}(J) \geq 1 - 10\beta'$. Since $J = J_4(G)$ is a clique complex, we deduce $d_{(2,2,0)}(J) = \mathbb{P}(e \in J_{(2,2,0)}) > 1/11$. But this contradicts $(2, 2, 0) \notin L$.

In either case we obtain a contradiction to the existence of the divisibility barrier \mathcal{P} . \square

Now we consider divisibility barriers with four parts.

Lemma 8.16. *Let $1/n \ll c \ll \mu \ll 1$, and G be a 3-graph on n vertices with $\delta(G) \geq (3/4 - c)n$. Write $J = J_4(G)$ for the clique 4-complex of G . Then there is no partition \mathcal{P} of $V(G)$ into four parts of size at least $n/4 - \mu n$ such that $L_{\mathcal{P}}^{\mu}(J_4)$ is minimal and incomplete.*

Proof. We introduce constants μ', β, β' with $\mu \ll \mu' \ll \beta \ll \beta' \ll 1$. Suppose for a contradiction we have a partition \mathcal{P} of V into parts V_1, V_2, V_3, V_4 of size at least $n/4 - \mu n$ such that $L = L_{\mathcal{P}}^{\mu}(J_4)$ is incomplete and minimal. Note that all parts have size at most $n/4 + 3\mu n$. Recall that $\delta(J) \geq (n, n-1, (3/4-c)n, (1/4-3c)n)$ by (7). Now we need the following claim.

Claim 8.17.

- (i) If $\mathbf{i} \in \mathbb{Z}^4$ with $|\mathbf{i}| = 4$ and $d_{\mathbf{i}}(J) \geq \beta$ then $d_{\mathbf{i}}(J) \geq 1 - \beta$.
- (ii) If $\mathbf{i} \in \mathbb{Z}^4$ with $|\mathbf{i}| = 3$ and $d_{\mathbf{i}}(J) \geq 2\beta$ then $d_{\mathbf{i}}(J) \geq 1 - \beta$.

To prove the claim, first consider any $\mathbf{i} \in \mathbb{Z}^4$ with $|\mathbf{i}| = 4$ and $d_{\mathbf{i}}(J) \geq \beta$. Note that $\mathbf{i} \in L$. By Proposition 8.7(iv) there is some $\mathbf{i}' = \mathbf{i} - \mathbf{u}_j \in \partial \mathbf{i}$ such that $d_{\mathbf{i}'}(J) \geq d_{\mathbf{i}}(J)^{3/4} + O(1/n)$. Let B be the set of edges $e' \in J_{\mathbf{i}'}$ that lie in at least $\mu'n$ edges $e \in J_4$ with $\mathbf{i}(e) \neq \mathbf{i}$. Then there is some $j' \neq j$ for which the number of edges with index vector $\mathbf{i} - \mathbf{u}_j + \mathbf{u}_{j'}$ is at least $|B|\mu'n/12$. This must be less than μn^4 , otherwise we have $\mathbf{i} - \mathbf{u}_j + \mathbf{u}_{j'} \in L$, so $\mathbf{u}_j - \mathbf{u}_{j'} \in L$, contradicting minimality. It follows that $d_{\mathbf{i}'}(B) \leq \mu'$ (say), so $d_{\mathbf{i}'}(J \setminus B) \geq d_{\mathbf{i}}(J)^{3/4} - 2\mu'$. Next we show that if $e' \in J \setminus B$ then $e' \cup \{x\} \in J_{\mathbf{i}}$ for all but $2\mu'n$ vertices $x \in V_j$. This holds because there are at least $\delta_3(J) \geq (1/4 - 3c)n$ vertices x such that $e' \cup \{x\} \in J_4$, all but at most $\mu'n$ of these lie in V_j (as $e' \notin B$), and all parts have size at most $n/4 + 3\mu n$. It follows that $d_{\mathbf{i}}(J) \geq (1 - 10\mu')d_{\mathbf{i}'}(J \setminus B) \geq (1 - 10\mu')(d_{\mathbf{i}}(J)^{3/4} - 2\mu')$. Since $d_{\mathbf{i}}(J) \geq \beta$ this implies $d_{\mathbf{i}}(J) \geq 1 - \beta$, so we have proved (i). For (ii), observe that by Proposition 8.7(ii) there must be some \mathbf{i}' such that $|\mathbf{i}'| = 4$, $\mathbf{i} \leq \mathbf{i}'$ and $d_{\mathbf{i}'}(J) \geq (3/4 - c)2\beta \geq \beta$. Then by (i) we have $d_{\mathbf{i}}(J) \geq 1 - \beta$, so by Proposition 8.7(i) we have $d_{\mathbf{i}'}(J) \geq d_{\mathbf{i}}(J) \geq 1 - \beta$. This completes the proof of Claim 8.17.

Returning to the proof of the lemma, we show next that $d_{\mathbf{i}}(J) \leq 2\beta$ whenever \mathbf{i} is a permutation of $(3, 0, 0, 0)$. For suppose this fails, say for $\mathbf{i} = (3, 0, 0, 0)$. Then Claim 8.17 gives $d_{(3,0,0,0)}(J) \geq 1 - \beta$. Now Proposition 8.7(iii) gives $d_{(4,0,0,0)}(J) \geq 1 - 4\beta$, so $(4, 0, 0, 0) \in L$. Next, Proposition 8.7(ii) for $\mathbf{i}' = (2, 0, 0, 0)$ gives $(3/4 - c)n \leq (n/4 + 3\mu n) \sum_{j \in [4]} d_{\mathbf{i}'+\mathbf{u}_j}(J)$, so without loss of generality $d_{(2,1,0,0)}(J) \geq 2\beta$. Then Claim 8.17 gives $d_{(2,1,0,0)}(J) \geq 1 - \beta$. Now Proposition 8.7(iii) gives $d_{(3,1,0,0)}(J) \geq 1 - 4\beta$, so $(3, 1, 0, 0) \in L$. But then $(4, 0, 0, 0) - (3, 1, 0, 0) = (1, -1, 0, 0) \in L$ contradicts minimality. Thus $d_{\mathbf{i}}(J) \leq 2\beta$ whenever \mathbf{i} is a permutation of $(3, 0, 0, 0)$.

Now returning to the above inequality $(3/4 - c)n \leq (n/4 + 3\mu n) \sum_{j \in [4]} d_{\mathbf{i}'+\mathbf{u}_j}(J)$, where \mathbf{i}' is any permutation of $(2, 0, 0, 0)$, we see that for any \mathbf{i}'' which is a permutation of $(2, 1, 0, 0)$ we have $d_{\mathbf{i}''}(J) \geq 2\beta$, and therefore $d_{\mathbf{i}''}(J) \geq (1 - \beta)$ by Claim 8.17. Also, applying Proposition 8.7(ii) to $(1, 1, 0, 0)$, we see that without loss of generality $d_{(1,1,1,0)}(J) \geq 2\beta$. Then Claim 8.17 gives $d_{(1,1,1,0)}(J) \geq 1 - \beta$. Now each of $d_{(2,2,0,0)}(J)$ and $d_{(2,1,1,0)}(J)$ is at least $1 - 4\beta$ by Proposition 8.7(iii). It follows that L contains $(2, 2, 0, 0)$, $(2, 1, 1, 0)$ and $(2, 2, 0, 0) - (2, 1, 1, 0) = (0, 1, -1, 0)$, contradicting minimality. Thus there is no such partition \mathcal{P} . \square

8.5. The main case of Theorem 1.1. As outlined earlier, Theorem 2.8 will imply the desired result, except for 3-graphs that are close to the extremal configuration. We start with a lemma that handles such 3-graphs. The proof requires a bound on the minimum vertex degree that guarantees a perfect matching in a 4-graph. We just quote the following

(slight weakening of a) result of Daykin and Häggkvist [8] that suffices for our purposes. (The tight bound was recently obtained by Khan [23].)

Theorem 8.18. *Let H be a 4-graph on n vertices, where $4 \mid n$, such that every vertex belongs to at least $\frac{3}{4}\binom{n}{3}$ edges. Then H contains a perfect matching.*

The following lemma will be used for 3-graphs that are close to the extremal configuration. Note that we will be able to satisfy the condition that $|V_1|$ is even by applying Lemma 8.10.

Lemma 8.19. *Suppose that $1/n \ll \gamma \ll c \ll 1$. Let V be a set of $4n$ vertices partitioned into V_1, V_2, V_3, V_4 each of size at least $n - \gamma n$. Suppose J is a 4-complex on V such that*

- (i) $d(J_4[V_j]) \geq 1 - \gamma$ for each $j \in \{2, 3, 4\}$,
- (ii) for each $\mathbf{i} \in (2, 1, 1, 0), (2, 1, 0, 1), (2, 0, 1, 1)$ we have $d_{\mathbf{i}}(J) \geq 1 - \gamma$,
- (iii) for every vertex $v \in V$ at least cn^4 edges of J_4 contain v and have an even number of vertices in V_1 , and
- (iv) $|V_1|$ is even.

Then J_4 contains a perfect matching.

Proof. We say that an edge of J_4 is *even* if it has an even number of vertices in V_1 , and *odd* otherwise. For each $j \in \{2, 3, 4\}$, we say that a vertex $v \in V_j$ is *good* if there are at least $(1 - 2\sqrt{\gamma})\binom{n}{3}$ edges of $J_4[V_j]$ which contain v . By (i) at most $\sqrt{\gamma}n$ vertices of each of V_2, V_3 and V_4 are bad; here we note that each part has size at most $n + 3\gamma n$. We say that a pair $u, v \in V_1$ is *good* if for each $\mathbf{i} \in \{(2, 0, 1, 1), (2, 1, 0, 1), (2, 1, 1, 0)\}$ there are at least $(1 - 2\sqrt{\gamma})n^2$ edges $e \in J_4$ with $u, v \in e$ and $\mathbf{i}(e) = \mathbf{i}$. For each such \mathbf{i} , by (ii) at most $\sqrt{\gamma}\binom{n}{2}$ pairs $u, v \in V_1$ do not have this property, and so at most $3\sqrt{\gamma}\binom{n}{2}$ pairs $u, v \in V_1$ are bad. We say that a vertex $u \in V_1$ is *good* if it lies in at least $(1 - 2\gamma^{1/4})n$ good pairs $u, v \in V_1$. Then at most $3\gamma^{1/4}n$ vertices of V_1 are bad. In total, the number of bad vertices is at most $\gamma^{1/5}n$, say.

Now let E be a maximal matching in J such that every edge in E has an even number of vertices in V_1 and contains a bad vertex. We claim that E covers all bad vertices. For suppose some bad vertex v is not covered by E . Since each edge in E contains a bad vertex, at most $4\gamma^{1/5}n$ vertices are covered by E , so at most $4\gamma^{1/5}n^4$ edges of J_4 contain a vertex covered by E . Then by (iii) we may choose an edge $e \in J_4$ which contains v , has an even number of vertices in V_1 and doesn't contain any vertex covered by E , contradicting maximality of E . So E must cover every bad vertex of J . We will include E in our final matching and so we delete the vertices it covers. We also delete the vertices covered by another matching E' of at most 16 edges, disjoint from E , so as to leave parts V'_1, V'_2, V'_3, V'_4 such that 8 divides $|V'_1|$ and 4 divides each of $|V'_2|, |V'_3|, |V'_4|$. The edges in E' will have index $(2, 1, 1, 0), (2, 1, 0, 1)$ or $(2, 0, 1, 1)$. Note that by (ii) we can greedily choose a matching disjoint from E containing 16 edges of each of these indices, so we only need to decide how many of these we want to include in E' . First we arrange that there are an even number of remaining vertices in each part. Note that $|V|$ and $|V_1|$ are even, and the number of vertices remaining in V_1 is still even, as we only used edges with an even number of vertices in V_1 . Then an even number of parts have an odd number of vertices remaining (after the deletion of E). If there are two such parts, say V_2 and V_4 , then we remedy this by including an edge of index $(2, 1, 0, 1)$ in E' , so we can assume there are no such parts. Now there are an even number of parts in which the number of vertices remaining is not divisible by 4. If V_1 is one of these parts we include an edge of each index $(2, 1, 1, 0), (2, 1, 0, 1), (2, 0, 1, 1)$ in E' ; then

each part has an even number of remaining vertices and the number in V_1 is divisible by 4. There may still be two parts where the number of vertices remaining is not divisible by 4. If so, say they are V_3 and V_4 , we include 2 edges of index $(2, 0, 1, 1)$ in E' . Thus we arrange that the number of vertices remaining in each part is divisible by 4. Finally, if the number of remaining vertices in V_1 is not divisible by 8 then we include 2 more edges of each index in E' . Thus we obtain E' with the desired properties. We delete the vertices of $E \cup E'$ and let V'_1, V'_2, V'_3, V'_4 denote the remaining parts.

Next we choose a perfect matching M_1 of good pairs in V'_1 . This is possible because $|V'_1|$ is even, and any vertex in V'_1 is good, in that it was incident to at least $(1 - 2\gamma^{1/4})n$ good pairs in V_1 , so is still incident to at least $n/2$ good pairs in V'_1 . Now we greedily construct a matching in J_4 , by considering each pair in M_1 in turn and choosing an edge of index $(2, 1, 1, 0)$ containing that pair. There are at most $n/2$ pairs in M_1 , so when we consider any pair uv in M_1 , at least $n/3$ vertices in each of V'_2 and V'_3 are still available, in that they have not already been selected when we considered some previous pair in M_1 . Since uv is good, at most $3\sqrt{\gamma}n^2 < (n/3)^2$ of these pairs do not form an edge with uv , so we can form an edge as required. Thus we construct a matching E'' that covers V'_1 . We delete the vertices of E'' and let V''_2, V''_3, V''_4 denote the remaining parts. Note that the number of vertices used by E'' in each of V'_2 and V'_3 is $|V'_1|/2$, which is divisible by 4. Since 4 divides each of $|V'_2|, |V'_3|, |V'_4|$, it also divides each of $|V''_2|, |V''_3|, |V''_4|$. Furthermore, for any $j \in \{2, 3, 4\}$ and $x \in V_j$, since x is good, the number of edges of $J_4[V''_j]$ containing x is at least $\binom{|V''_j|}{3} - 3\sqrt{\gamma}\binom{n}{3} > \frac{3}{4}\binom{|V''_j|}{3}$, so $J_4[V''_j]$ contains a perfect matching by Theorem 8.18. Combining these matchings with E, E' and E'' we obtain a perfect matching in J_4 . \square

We can now give the proof of Theorem 1.1. Suppose that G is a 3-graph on n vertices, where n is sufficiently large and divisible by 4. For now we assume $\delta(G) \geq 3n/4 - 1$, postponing the case that $8 \mid n$ and $\delta(G) = 3n/4 - 2$ to the final subsection. We introduce constants α, β with $1/n \ll \alpha \ll \beta \ll 1$. We will show that G has a perfect tetrahedron packing; equivalently, that J_4 has a perfect matching, where $J = J_4(G)$ is the clique 4-complex of G . By (7) we have $\delta(J) \geq (n, n-1, 3n/4-2, n/4-3)$, so we can apply Theorem 2.8. Supposing that J_4 does not contain a perfect matching, we conclude that there is a space barrier or a divisibility barrier. As in the proof of Theorem 8.2 there are no space barriers by Propositions 8.3 and 8.6, so we may choose a minimal divisibility barrier, i.e. a partition \mathcal{Q} of $V(G)$ into parts of size at most $n/4 - \mu n$ such that $L_{\mathcal{Q}}^{\mu}(J_4)$ is minimal and incomplete. We have excluded all but one possibility for \mathcal{Q} . Indeed, \mathcal{Q} cannot have more than two parts by Lemmas 8.14 and 8.16, so must have two parts. By Lemmas 8.8 and 8.9, we deduce that \mathcal{Q} partitions $V(G)$ into parts U and V of size at least $n/4 - \alpha n$ such that at most αn^4 edges of J_4 have an odd number of vertices in each part.

By Lemma 8.12, there is a partition \mathcal{P} of $V(G)$ into parts V_1, V_2, V_3 and V_4 each of size at least $n/4 - \beta n$, so that $V_1 = U$ (without loss of generality) and $d_i(G) \geq 1 - \beta$ for each good \mathbf{i} (recall that good index vectors are those that appear in the extremal example). To apply Lemma 8.19 we also need to arrange that $|V_1|$ is even, and that every vertex is in many even edges of J_4 , where we say that an edge of J_4 is odd or even according to the parity of its intersection with V_1 . We say that a vertex is *good* if it lies in fewer than $n^3/400$ odd edges of J_4 (and *bad* otherwise). Then at most $1600\alpha n$ vertices are bad, since J_4 has at most αn^4 odd edges. We let V'_1 consist of all good vertices in V_1 and all bad vertices in $V_2 \cup V_3 \cup V_4$, let V'_2 consist of all good vertices in V_2 and all bad vertices in V_1 , and let V'_3 and V'_4 consist of

all good vertices in V_3 and V_4 respectively. Note that any vertex that has a different index with respect to the partition \mathcal{P}' of $V(G)$ into the V'_j than with respect to \mathcal{P} is bad, so we have $|V'_1| = n/4 \pm 2\beta n$. So by Lemma 8.10, there exists a copy of K_4^3 in G (i.e. an edge of J_4) with an odd number of vertices in V'_1 . If $|V'_1|$ is odd, we choose one such edge of J_4 , and delete the vertices of this edge from V'_1, V'_2, V'_3, V'_4 , and J . Thus we can arrange that $|V'_1|$ is even.

Having changed the index of at most $1600\alpha n$ vertices of J and deleted at most four vertices, we have $|V'_i| \geq n/4 - 2\beta n$ for each $i \in [4]$, and $d_i(G) \geq 1 - 2\beta$ for each good i with respect to the partition into V'_1, V'_2, V'_3, V'_4 . It remains to show that any undeleted vertex v belongs to many edges with an even number of vertices in V'_1 . To avoid confusion, we now use the terms X -odd and X -even, where X is V_1 or V'_1 , to describe edges according to the parity of their intersection with X . First suppose that v is good. Then any edge of J_4 containing v that is V'_1 -odd but not V_1 -odd must contain a bad vertex, so there are at most $1600\alpha n^3$ such edges. Since v belongs to at most $n^3/400$ V_1 -odd edges of J_4 , it belongs to at most $n^3/400 + 1600\alpha n^3$ V'_1 -odd edges of J_4 . Also, v belongs to at least $n^3/200$ edges of J_4 , by the lower bound on $\delta(J)$, so v belongs to at least $n^3/500$ V'_1 -even edges. Now suppose that v is bad. Then any V_1 -odd edge of J_4 containing v and no other bad or deleted vertices is V'_1 -even. The number of such edges is at least $n^3/400 - 1600\alpha n^3 - 4n^2 \geq n^3/500$. This shows that every vertex is contained in at least $n^3/500$ V'_1 -even edges of J_4 .

Finally, Lemma 8.19 implies that J_4 has a perfect matching.

8.6. The case when 8 divides n . It remains to consider the case when $8 \mid n$ and $\delta(G) = 3n/4 - 2$. We apply the same proof as in the previous case, noting that we only needed $\delta(G) \geq 3n/4 - 1$ if $|V'_1|$ was odd, when we used it to find a V'_1 -odd tetrahedron. Examining the proof of Proposition 8.10, we see that a V'_1 -odd tetrahedron exists under the assumption $\delta(G) = 3n/4 - 2$, unless we have $n/4 - 2 \leq |V'_1| \leq n/4$. Furthermore, we only need a V'_1 -odd tetrahedron when $|V'_1|$ is odd, so we only need to consider the exceptional case that $|V'_1| = n/4 - 1$ and there is no V'_1 -odd tetrahedron.

Let c_1 satisfy $\beta \ll c_1 \ll 1$. Recall that $d_i(G) \geq 1 - 2\beta$ for all good i . Note that for each bad triple e there is a part V'_j such that for every $x \in V'_j \setminus e$, $e \cup \{x\}$ is V'_1 -odd, and every triple in $e \cup \{x\}$ apart from e is good. (We describe triples as good or bad according to their index vectors.) If e is an edge, then some such triple is not an edge. Since there is no V'_1 -odd tetrahedron, there are at most βn^3 bad edges by Proposition 8.7(iii). For any $2/n < c < 1/4$, we say that a pair is c -bad if it is contained in at most $(3/4 - 2c)n$ good edges. Since $\delta(G) = 3n/4 - 2$, any c -bad pair is contained in at least cn bad edges. If $\beta \ll c$ then there are at most cn^2 c -bad pairs. Since there is no V'_1 -odd tetrahedron, any bad edge of G contains at least one $1/30$ -bad pair.

Without loss of generality we have $|V'_4| \geq n/4 + 1$. Thus for each $v_2 \in V'_2$ and $v_3 \in V'_3$ there is some $v_4 \in V'_4$ with $v_2v_3v_4 \in G$. Each such edge is bad, so contains a $1/30$ -bad pair. The bad pair can only be v_2v_3 for at most c_1n^2 such triples. Thus without loss of generality there is a vertex $v \in V_4$ such that at least $n/10$ pairs v_3v with $v_3 \in V'_3$ are $1/30$ -bad. For each such pair there are at least $n/30$ vertices $v_2 \in V'_2$ such that v_2v_3v is a bad edge. For each such bad edge and each $v_1 \in V'_1$, some triple in $v_1v_2v_3v$ is not an edge. Since $d_{(1,1,1,0)}(G) \geq 1 - 2\beta$, there are at most c_1n vertices $v_1 \in V'_1$ such that $v_1v_2v_3$ is not an edge for at least c_1n^2 pairs v_2v_3 . Thus for all but at most c_1n vertices $v_1 \in V'_1$ there are at least $n^2/300 - c_1n^2$ pairs

v_2v_3 such that one of v_1v_2v or v_1v_3v is not an edge. Then v_1v is $1/200$ -bad, so there are at least $n/200$ vertices $v_4 \in V'_4$ for which v_1vv_4 is a (bad) edge.

Now for each $v_1 \in V'_1$, $v_4 \in V'_4$ such that v_1vv_4 is an edge and $x \in V'_2 \cup V'_3$, at least one of v_1xv_4 , v_1xv , xvv_4 is not an edge. Since $d_{(1,1,0,1)}(G)$ and $d_{(1,1,0,1)}(G)$ are at least $1 - 2\beta$, there are at most c_1n vertices $v_1 \in V'_1$ such that v_1xv_4 is not an edge for at least c_1n^2 pairs xv_4 . For $v_1 \in V'_1$, let $N_4(v_1)$ be the set of vertices $v_4 \in V'_4$ such that v_1vv_4 is an edge, and v_1xv_4 is an edge for all but at most $600c_1n$ vertices $x \in V'_2 \cup V'_3$. Let V''_1 be the set of vertices $v_1 \in V'_1$ such that $|N_4(v_1)| \geq n/300$. Then $|V'_1 \setminus V''_1| \leq 2c_1n$, otherwise there are at least c_1n vertices $v_1 \in V'_1 \setminus V''_1$ such that there are at least $n/200 - |N_4(v_1)| \geq n/600$ vertices $v_4 \in V'_4$ such that v_1vv_4 is an edge and $v_4 \notin N_4(v_1)$, so v_1xv_4 is not an edge for at least $600c_1n \cdot n/600 = c_1n^2$ pairs xv_4 , which is a contradiction.

For each $v_1 \in V''_1$, $v_4 \in N_4(v_1)$ and all but at most $600c_1n$ vertices $x \in V'_2 \cup V'_3$, v_1xv_4 is an edge, so at least one of v_1xv and xvv_4 is not an edge. Since $d(v_1v) + d(vv_4) \geq 2\delta(G) = 3n/2 - 4$, and $|V'_j| \geq n/4 - 2\beta n$ for each $j \in [4]$, precisely one of v_1xv and xvv_4 is not an edge for all but at most $2000c_1n$ vertices $x \in V'_2 \cup V'_3$. Also, v_1xv and xvv_4 are edges for all but at most $2000c_1n$ vertices $x \in V'_1 \cup V'_4$. Now there can be at most c_1n vertices $v_1 \in V''_1$ such that v_1v_2v is an edge for at least $3000c_1n$ vertices $v_2 \in V'_2$. Otherwise, since we have at least $n/300$ choices of $v_4 \in N_4(v_1)$, so v_1vv_4 is an edge, and $1000c_1n$ choices of v_2 such that v_2vv_4 is an edge, there must be at least $c_1^2n^3$ good triples $v_1v_2v_4$ that are not edges, contradicting $d_{(1,1,0,1)}(G) \geq 1 - 2\beta$. Similarly, there are at most c_1n vertices $v_1 \in V''_1$ such that v_1v_3v is an edge for at least $3000c_1n$ vertices $v_3 \in V'_3$. But then we can choose $v_1 \in V''_1$ such that v_1xv is an edge for at most $6000c_1n$ vertices $x \in V'_2 \cup V'_3$, which contradicts $\delta(G) = 3n/4 - 2$.

Thus the exceptional case considered here cannot occur. The rest of the proof goes through as in the previous section, so this proves Theorem 1.1. \square

8.7. Strong stability for perfect matchings. We conclude this section by applying a similar argument to that in Lemma 8.19 to prove the stronger form of Theorem 2.10 alluded to earlier. For this we need the following lemma, a partite analogue of a theorem from [8], which gives a (very weak) bound on the vertex degree which guarantees a perfect matching in a k -partite k -system.

Lemma 8.20. *For any k there exists $d_k > 0$ such that if $1/n \ll d_k$ and G is a k -partite k -graph on vertex classes V_1, \dots, V_k of n vertices in which every vertex of G lies in at least $(1 - d_k)n^{k-1}$ edges of G , then G contains a perfect matching.*

Proof. We proceed by induction on k . The case $k = 2$ is immediate from Hall's theorem. Now suppose $k > 2$ and the result holds for $(k - 1)$ -graphs, for some $d_{k-1} > 0$. Introduce new constants with $1/n \ll d_k \ll d' \ll b \ll d_{k-1}, 1/k$, and let H be the $(k - 1)$ -partite $(k - 1)$ -graph on vertex classes V_1, \dots, V_{k-1} whose edges are all those $(k - 1)$ -tuples $(v_j)_{j=1}^{k-1}$ with $v_j \in V_j$ for $j \in [k - 1]$ which are contained in at least $(1 - b/4)n$ edges of G . Then H has at least $(1 - 4d_k/b)n^{k-1}$ edges, and so at most $d'n$ vertices $v \in V_1 \cup \dots \cup V_{k-1}$ lie in fewer than $(1 - d')n^{k-2}$ edges of H . Choose greedily a matching M_0 in G of size at most $d'n$ which covers every such vertex, and form G' , H' and V'_j for $j \in [k]$ by deleting the at most $kd'n$ vertices covered by M_0 from G , H and V_j for $j \in [k]$. So each V'_j has equal size $n_1 \geq (1 - d')n$.

Now, let T be a multiset of bn_1 $(k-1)$ -tuples $(v_j)_{j=1}^{k-1}$ with $v_j \in V'_j$ for $j \in [k-1]$ chosen uniformly at random and independently of each other choice (with repetition of choice allowed). Then, counting repetitions, we have the following properties.

- (i) With probability $1 - o(1)$, for any vertex $v \in V'_k$ there are at least $bn_1/2$ sets $S \in T$ such that $S \cup \{v\} \in G'$. Indeed, this follows from the Chernoff bound (Lemma 6.13), since for any $v \in V'_k$ the number of $(k-1)$ -tuples $S = (v_j)_{j=1}^{k-1}$ with $v_j \in V'_j$ for $j \in [k-1]$ and $S \cup \{v\} \in G'$ is at least $(1 - d_k)n^{k-1} - kd'n^{k-1} \geq 2n_1^{k-1}/3$.
- (ii) With probability $1 - o(1)$, all but at most b^2n_1 sets $S \in T$ are edges of H' . Indeed, this also follows from the Chernoff bound, since $|H'| \geq |H| - kd'n^{k-1} \geq (1 - b/2)n^{k-1}$.
- (iii) With probability at least $1/2$, at most $2b^2kn_1$ sets $S \in T$ intersect another member of T . Indeed, the probability that a vertex $v \in V'_1 \cup \dots \cup V'_{k-1}$ is covered by T is $1 - (1 - 1/n_1)^{bn_1} \geq b(1 - b)$, and so $\mathbb{E}|V(T)| \geq b(1 - b)kn_1$. But $|V(T)| \leq bkn_1$ in any case, so with probability at least $1/2$ we have $|V(T)| \geq b(1 - 2b)kn_1$, from which it follows that at most $2b^2kn_1$ sets $S \in T$ intersect another member of T .

Fix an outcome of the random choice of T so that each of these events holds, and let $M_1^H \subseteq T$ consist of all edges of $T \cap H'$ which do not intersect any other edge of T (so M_1^H is a matching in H'). Then by (i), (ii) and (iii), for any $v \in V'_k$ there are at least $bn_1/2 - b^2n_1 - 2b^2kn_1 \geq bn_1/3$ sets $S \in M_1^H$ for which $S \cup \{v\} \in G'$.

Let H'' be the subgraph of H' formed by deleting the vertices covered by M_1^H ; then the vertex classes V''_1, \dots, V''_k of H'' have equal size at least $n_1 - bn_1 \geq n - 2bn$. In particular, every vertex of H'' lies in at least $(1 - d')n^{k-2} - 2bkn^{k-2} \geq (1 - d_{k-1})n^{k-2}$ edges of H'' . So by our inductive assumption, H'' contains a perfect matching M_2^H . Let $M^H = M_1^H \cup M_2^H$; then M^H is a perfect matching in H' . Let B be the bipartite graph whose vertex classes are M^H and V'_k in which there is an edge between $S \in M^H$ and $v \in V'_k$ if $S \cup \{v\} \in G'$. Then by definition of H , and the fact that at most $d'n$ vertices of V_k were deleted in forming V'_k , every $S \in M^H$ lies in at least $(1 - b/3)n_1$ edges of B . Also, by choice of M_1^H every $v \in V'_k$ lies in at least $bn_1/3$ edges of B . So by Hall's Theorem there is a perfect matching in B . The pairs (v, S) of this matching form a perfect matching in G' ; combined with M_0 this gives a perfect matching in G . \square

We can now present our strong stability theorem for perfect matchings, showing that any k -graph with minimum degree sufficiently close to $n/2$ has the same structure as one of the extremal examples. Note that the assumption $k \geq 3$ is necessary for this conclusion, but for $k = 2$ we still have the usual stability result that G is approximately contained in an extremal example (this follows from Theorem 7.11).

Theorem 8.21. *Suppose that $1/n \ll c \ll 1/k$, $k \geq 3$ and $k|n$, and let G be a k -graph on n vertices with $\delta(G) \geq (1/2 - c)n$ and no perfect matching. Then there is a partition of $V(G)$ into parts V_1, V_2 of size at least $\delta(G)$ and $a \in \{0, 1\}$ so that $|e \cap V_1| = a \pmod 2$ for all edges e of G .*

Proof. Let $1/n \ll b \ll c \ll 1/k$. By Theorem 2.10, there is a partition of $V(G)$ into parts V_1, V_2 of size at least $\delta(G)$ and $a \in \{0, 1\}$ so that all but at most bn^k edges $e \in G$ have $|e \cap V_1| = a \pmod 2$. Say that an edge e is *good* if $|e \cap V_1| = a \pmod 2$. We will show that all edges of G are good. Suppose for a contradiction that e_0 is a bad edge. We say that a vertex is *bad* if it is contained in at least $k\sqrt{bn^{k-1}}$ bad edges. Then there are at most \sqrt{bn} bad vertices. Since $\delta(G) \geq (1/3 + c)n$ we can greedily choose a matching E that covers all

bad vertices and is disjoint from e_0 . We let V'_1 and V'_2 be the remaining vertices in V_1 and V_2 after removing the vertices covered by E , and possibly also e_0 . By choosing whether or not to remove e_0 we may ensure that $|V'_1|$ is even if $a = 0$, and has the same parity as n/k if $a = 1$.

Let $I = \{0 \leq i \leq k : i \equiv a \pmod{2}\}$. We next choose numbers $n_i \geq cn$ for $i \in I$, such that $\sum_{i \in I} n_i = n/k$ and $\sum_{i \in I} in_i = |V'_1|$. To see that this is possible we use a variational argument: we start with all n_i close to cn apart from n_0 or n_1 (according as $0 \in I$ or $1 \in I$), which is chosen so that $\sum_i n_i = n/k$, so $\sum_{i \in I} in_i$ initially is at most $n/k + k^2cn \leq |V'_1|$, and has the same parity as $|V'_1|$. We repeatedly decrease some n_i by 1 and increase n_{i+2} by 1; this increases $\sum_{i \in I} in_i$ by 2, so we eventually achieve $\sum_i in_i = |V'_1|$, as required. Now we choose partitions \mathcal{P}_1 of V'_1 and \mathcal{P}_2 of V'_2 uniformly at random from those such that \mathcal{P}_1 has n_i parts of size i and \mathcal{P}_2 has n_i parts of size $k - i$ for each i . For each $i \in I$, let $X_1^i, \dots, X_{n_i}^i$ be the parts in \mathcal{P}_1 of size i , and $Z_1^i, \dots, Z_{n_i}^i$ be the parts in \mathcal{P}_2 of size $k - i$. Then we may form a partition of $V(G)$ into parts Y_j^i for $i \in I$ and $j \in [k]$ uniformly at random, where for each i and j the part Y_j^i contains one vertex from each X_α^i if $j \leq i$, and one vertex from each Z_α^i if $i + 1 \leq j$. So each Y_j^i has size $n_i \geq cn$. We consider for each $i \in I$ an auxiliary k -partite k -graph H^i on vertex classes Y_1^i, \dots, Y_k^i whose edges are those k -partite k -tuples that are edges of G . So the H^i are vertex-disjoint, and to find a perfect matching in G , it suffices to show that with high probability each H^i has a perfect matching.

Fix some $i \in I$. Since we covered all bad vertices by E , all vertices in H^i belong to at most $k\sqrt{bn}^{k-1}$ bad edges. Now fix $y_1 \in Y_1^i$. Say that a $(k-2)$ -tuple $(y_j)_{j=2}^{k-1}$ with $y_j \in Y_j^i$ for $j \in \{2, \dots, k-1\}$ is y_1 -bad if $y_1 \dots y_{k-1}$ belongs to more than $kb^{1/4}n$ bad edges. Then there are at most $b^{1/4}n^{k-2}$ y_1 -bad $(k-2)$ -tuples. Suppose that $i < k$, so $Y_k^i \subseteq V_2$; then a k -tuple $y_1 \dots y_{k-1}x$ is good if $x \in V_2$ or bad if $x \in V_1$. Since $\delta(G) \geq (1/2 - c)n$ and $|V_2| \leq (1/2 + c)n$, if $(y_i)_{i=2}^{k-1}$ is y_1 -good then $y_1 \dots y_{k-1}x$ is an edge for all but at most $3cn$ vertices $x \in V_2$. The number of these vertices x that lie in Y_k^i is hypergeometric with mean at least $(1 - 3c)|Y_k^i|$, where $|Y_k^i| = n_i \geq cn$. So by the Chernoff bound (Lemma 6.13), with probability $1 - o(1)$ there are at least $(1 - 4c)n_i$ such vertices x for any choice of $y_1 \in Y_1^i$ and y_1 -good $(y_i)_{i=2}^{k-1}$. Thus y_1 is contained in at least $(n_i^{k-2} - b^{1/4}n^{k-2})(1 - 4c)n_i > (1 - 5c)n_i^{k-1}$ edges of H^i . By symmetry, with positive probability the same bound holds for all vertices in any H^i (for $i = k$ we have $Y_k \subseteq V_1$, in which case we proceed similarly with the roles of V_1 and V_2 switched). Then each H^i has a perfect matching by Lemma 8.20. This contradicts our assumption that G does not have a perfect matching. Thus the assumption that there is a bad edge e_0 was false, so all edges are good. \square

9. THE GENERAL THEORY

In this final section, we give a more general result, which epitomises the geometric nature of the theory, in that it almost entirely dispenses with degree assumptions. We cannot hope to avoid such assumptions entirely when using methods based on hypergraph regularity, as sparse hypergraphs may have empty reduced systems. Our degree assumptions are as weak as possible within this context, in that the proportional degrees can be $o(1)$ as $n \rightarrow \infty$. The hypotheses of our result are framed in terms of the reduced system provided by hypergraph regularity, so it takes a while to set up the statement, and it is not as clean as that of our main theorems (which is why we have deferred the statement until now). However, the extra

generality provided by the geometric context will be important for future applications, even in the context of minimum degree thresholds for hypergraph packing problems, where the tight minimum degree may not imply the minimum degree sequence required by our main theorems. (We intend to return to this point in a future paper.) In an attempt to avoid too much generality, we will restrict attention to the non-partite setting here.

First we describe the setting for our theorem. We start with the hypergraph regularity decomposition.

Regularity setting. Let J be a k -complex on n vertices, where $k \mid n$. Let \mathcal{Q} be a balanced partition of $V(J)$ into h parts, and J' be the k -complex of \mathcal{Q} -partite edges in J . Let P be an a -bounded ε -regular vertex-equitable \mathcal{Q} -partition $(k-1)$ -complex on $V(J)$, with clusters V_1, \dots, V_{m_1} of size n_1 . Let G be a \mathcal{Q} -partite k -graph on $V(J)$ that is ν -close to J'_k and perfectly ε -regular with respect to P . Let $Z = G \triangle J'_k$.

Next we describe the setting for the reduced system, which names the constructions that were used in the proof of Theorem 7.11. While we do not specify the source of the subsystem (R_0, M_0) , it is perhaps helpful to think of it as being randomly chosen so as to inherit the properties of (R, M) , as in the proof of Lemma 5.4.

Reduced system setting. Let $R^1 := R_{PQ}^{J'Z}(\nu, \mathbf{c})$ on $[m_1]$. Also let (R, M) be a matched k -system on $[m]$, where R is the restriction of R^1 to $[m]$. Given any R' and M' which can be formed from R and M respectively by the deletion of the vertices of at most ψm edges of M , and any vertices $u, v \in V(R')$, choose $M_0 \subseteq M'$ such that $V_0 := V(M_0)$ satisfies $m_0 := |V_0| \leq m_0^*$ and M_0 includes the edges of M' containing u, v . Let $R_0 = R[V_0]$, $X = X(R_0, M_0) = \{\chi(e) - \chi(e') : e \in R_0, e' \in M_0\} \subseteq \mathbb{R}^{m_0}$ and $\Pi = \{\mathbf{x} \in \mathbb{Z}^{m_0} : \mathbf{x} \cdot \mathbf{1} = 0\}$.

Now we can state our general theorem (in the above setting).

Theorem 9.1. *Let $1/n \ll \varepsilon \ll 1/a \ll \nu, 1/h \ll c_k \ll \dots \ll c_1 \ll \psi \ll \psi' \ll \delta, 1/m_0^* \ll \alpha \ll \gamma, 1/k$. Suppose that $m \geq (1 - \psi)m_1$, and the following conditions hold.*

- (i) *For any R', M', u and v given as above, we may choose M_0 so that (R_0, M_0) satisfies $B(\mathbf{0}, \delta) \cap \Pi \subseteq CH(X)$.*
- (ii) *$\delta^+(D_\ell(R, M)) \geq \gamma m$.*
- (iii) *$L_P(R)$ is complete for any partition \mathcal{P} of $V(R)$ into parts of size at least $(\gamma - \alpha)m$.*
- (iv) *Every vertex of J is contained in at least γn^{k-1} edges of J .*

Then J_k contains a perfect matching.

Proof. We follow the proof of Theorem 7.11, outlining the modifications. Introduce constants with

$$\begin{aligned} 1/n \ll \varepsilon \ll d^* \ll d_a \ll 1/a \ll \nu, 1/h \ll d, c \ll c_k \ll \dots \ll c_1 \\ \ll \psi \ll 1/C' \ll 1/B, 1/C \ll \delta, 1/m_0^* \ll \alpha \ll \alpha' \ll \gamma, 1/k. \end{aligned}$$

Our set-up already provides $\mathcal{Q}, J', P, G, Z, R, M$, as in that proof. (Note that we are working in the non-partite setting, so there is no \mathcal{P}' or F . We also do not need to consider two reduced systems.) There is no Claim 7.12 as we already have M . The proof of Claim 7.13 is the same, except that for (iii), instead of the minimum F -degree assumption, we apply Lemma 7.14 to the edges of J containing v , using the assumption that every vertex is contained in at least γn^{k-1} edges of J . To prove Claim 7.15, instead of Lemma 5.6, we first apply Lemma 4.1 to see that (R_0, M_0) is (B, C) -irreducible for any u and v . It follows

that (R', M') is (B, C) -irreducible, then we apply Lemma 4.10 to see that $D_{C'}(R', M')$ is complete. The remainder of the proof only uses these claims, so goes through as before. \square

We conclude with a remark comparing our techniques to the absorbing method, which has been successfully applied to many hypergraph matching problems. The idea of this method is to randomly select a small matching M_0 in a k -graph G , and show that with high probability it has the property that it can ‘absorb’ any small set of vertices V_0 , in that there is a matching covering $V(M_0) \cup V_0$. Given such a matching M_0 , to find a perfect matching in G , it suffices to find an almost perfect matching in $G \setminus V(M_0)$, which is often a much simpler problem. The advantage of this method is that it avoids hypergraph regularity, and so leads to shorter proofs, when it works. However, the existence of an absorbing matching seems to be a fortuitous circumstance in each application of the method, and it is not clear how one could hope to find it in general problems. By contrast, our theory explains ‘why’ there is a perfect matching, by analysing the only possible obstructions (space barriers and divisibility barriers). This has the additional advantage of giving structural information, so we can apply the stability method to obtain exact results.

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